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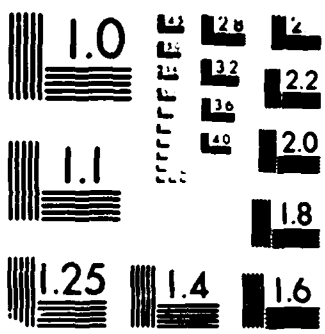

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RANDOM PACKING AND RANDOM COVERING SEQUENCES

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RANDOM PACKING AND RANDOM COVERING SEQUENCES

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In a sequential packing problem, random objects are uniformly and independently selected from some space. A selected object is either packed or rejected, depending on the distance between it and the nearest object which has been previously packed. A saturated packing is said to exist when it is no longer possible to pack any additional selections. The random packing density is the average proportion of the space which is occupied by the packed objects at saturation.

Results concerning the time of the first rejection in a packing sequence are given in Chapter 1. The accuracy of some common approximation formulas is investigated for several settings. The problems considered may be thought of as generalizations of the classical birthday problem.

Exact results concerning random packing densities are generally known only for some packing sequences in one-dimensional spaces. In Chapter 2, the packing densities of various computer generated codes are examined. These stochastically constructed codes provide a convenient way to study packing in multidimensional spaces. Asymptotic approximation formulas are given for the packing densities which arise from several different coding schemes. In one special case considered, a new method is found for approximating a planar density. The result obtained agrees closely with estimates obtained by others.

In Chapter 3 the distribution of the number of random selections needed to achieve a saturated packing is considered. In each of the settings examined, the results are compared with analogous results from an associated random covering problem.

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Chapter 0

Introduction

Sequential packing and covering problems have been investigated by numerous authors; however, there are quite a few interesting questions which have remained unanswered. Here some stopping rules arising in various packing and covering sequences are examined, and some fresh results concerning their distributions are presented.

The basic packing problem can be briefly described as follows. Random objects are sequentially selected from some space. A selected object is considered packed if it does not overlap any of the previously packed objects, otherwise the object is rejected. The selection process continues until it is no longer possible to pack any additional objects. The chief problem is to determine how much of the space is covered by the terminal collection of packed objects. So far, explicit solutions have been obtained only for packing sequences on one-dimensional spaces.

Chapter 1 examines the time of the first rejection in packing sequences. The problems considered, which will be called collision problems, may be viewed as generalizations of a familiar birthday problem where people are sequentially sampled at random until a matching of birthdays occurs. Applying the usual simplifying assumptions, this simple birthday setting may be modeled as follows. A sequence $C_1, C_2, \cdots$ of independent selections are made from the space $S = \{1, 2, \cdots, n\}$, where n represents the number of days in a year. The selections are made according to a uniform distribution, i.e.

$$P\{C_i = k\} = n^{-1}$$

for $1 \leq k \leq n$ and for each i. Letting τ count the number of selections needed to obtain

the first duplication of outcomes, it is well known that for fixed $t > 0$

$$P\left\{\frac{\tau}{\sqrt{2n}} > t\right\} \sim e^{-t^2} \quad \text{as } n \to \infty, \tag{0.1}$$

and (see [31]) that

$$E[\tau] \sim \sqrt{\frac{n\pi}{2}} \quad \text{as } n \to \infty. \tag{0.2}$$

A new result established in Chapter 1 yields, for this simple birthday setting, that

$$P\{\tau > [\![n^\alpha]\!]\} \sim \exp\left(-\frac{1}{2}n^{2\alpha-1}\right) \quad \text{as } n \to \infty \tag{0.3}$$

for $0 < \alpha < 2/3$. In other words (0.3) says that the common approximation formula (see [14])

$$P\{\tau > k\} \cong \exp(-k^2/2n), \tag{0.4}$$

which is suggested by (0.1), will hold reasonably well whenever $k < n^{2/3}$ and n is sufficiently large. A direct calculation in a related collision problem suggests that the equivalence stated in (0.3) does not hold for values of α which are greater than or equal to the upper bound 2/3.

Other settings for collision problems are also examined in Chapter 1. For the one-dimensional cases of packing arcs of equal length on the discrete and continuous circles, with a collision occurring if two arcs overlap, results similar to (0.1), (0.2), and (0.3) are obtained. However, for collision problems in higher dimensional spaces only analogs of (0.1) are proved exactly, with plausible arguments and simulation results used to provide support for the approximation

$$E[\tau] \cong \sqrt{\frac{\pi}{2p}} \tag{0.5}$$

which seems to hold in a large number of cases. In (0.5), p is used to denote the probability that two arbitrary random selections collide.

The time to the first collision is also investigated for several variations where arcs of unequal length are packed on the circle. These collision settings do not seem to have

been previously considered by others. For cases where the arc lengths are i.i.d. random variables, upper and lower bounds are obtained for $P\{\tau > k\}$, and results are given which indicate that the approximation formula (0.5) is not applicable for these settings.

Random packing densities for various random coding schemes are investigated in Chapter 2. Stochastically constructed codes provide a convenient way to study random packing in high-dimensional spaces, and a few special cases yield limiting values for packing densities which may be compared with analogous values obtained by others.

Itoh and Solomon [27] have studied the densities for cases where binary codewords are randomly packed by Hamming distance. They obtained approximation formulas for the densities in some of the cases which they investigated. In Section 2.2 a general approximation formula is proposed which not only does better than the approximations given in [27], it also does reasonably well for several additional cases. Letting n denote the total number of possible codewords and p be the probability that two arbitrary codewords collide, it is found that the random packing densities appear to be asymptotically approximated by a function of np which contains only one empirical constant.

Other sections in Chapter 2 examine random q-ary codes packed using various metrics. For the packing scheme discussed in Section 2.5, a two-dimensional analog of a one-dimensional result of Mackenzie [36] is obtained. This new formula yields an approximation for a planar packing density which is in close agreement with estimates determined in previous studies.

The total number of random selections required to achieve a saturated packing is examined in the final chapter. Chapter 3 also considers the total number of selections required to completely cover the space. In this variation, the random objects are allowed to overlap and none of the selections are rejected, and it is of interest to determine how many selections are required until each point of the space is covered by at least one object. A few such covering results have been found by Flatto and several others; however, the time to saturation does not seem to have been previously considered. It is found that the average number of selections required for saturation exceeds the average number required

for coverage in some settings, while the opposite is true in other cases. One easily proved, yet somewhat surprising result, is that for some packing sequences on continuous spaces the expected number of selections required for saturation is infinite.

Chapter 1

The Time to the Initial Collision of a Packing Sequence

1.1. Introduction

Let S be either a finite space, a k-dimensional unit torus ($k \in \{1, 2, \cdots\}$), or the surface of a sphere in E^k ($k \in \{2, 3, \cdots\}$), and suppose that points $C_1 \in S, C_2 \in S, \cdots$ are independently selected according to a uniform distribution. Thus if S is the finite space $\{s_1, s_2, \cdots, s_n\}$, then

$$P\{C_i = s_j\} = n^{-1}$$

for $i = 1, 2, \cdots$ and $1 \leq j \leq n$. If S is infinite then

$$P\{C_i \in B\} = \frac{|B|}{|S|} \quad (i = 1, 2, \cdots),$$

where $|\cdot|$ denotes Lebesgue measure and B is any measurable set.

Let μ be a metric on S. Then for some specified $\delta > 0$, two points C_i and C_j are said to collide provided that $\mu(C_i, C_j) < \delta$. If this condition is not met, then the points are said to be disjoint. A set of three or more points is said to be disjoint, or fairly packed, if the points are pairwise disjoint. In the literature, a collision is sometimes called a match or a coincidence.

Let $C_i \wedge C_j$ denote the event that C_i and C_j collide. If C_i and C_j are disjoint, write $C_i \vee C_j$. Denoting $\bigcup_{i=1}^{j}\{C_i\}$ by $\mathcal{C}(j)$, write $C_k \wedge \mathcal{C}(j)$ if C_k collides with at least one member of $\mathcal{C}(j)$. Let p denote the probability that two arbitrary points collide, i.e. $p = P\{C_i \wedge C_j\}$ ($i \neq j$). Note that p will always be positive.

Consider $C_1, C_2, \cdots$ chosen independently and uniformly from S. S, μ, P, δ and $\{C_i\}_{i=1}^{\infty}$ will collectively be called a packing sequence. Define a stopping time τ by

$$\{\tau = k\} = \{\#\mathcal{C}(k-1) = k-1,\ \mathcal{C}(k-1) \text{ is disjoint},\ C_k \wedge \mathcal{C}(k-1)\}.$$

Thus τ equals k if and only if C_k is involved in the first collision that occurs as the points $C_1, C_2, \cdots$ are selected one after another. Furthermore, if τ exceeds k, then the first k points of the sequence are pairwise disjoint. Using the terminology introduced above, it can also be said that the first k points have been fairly packed whenever τ exceeds k. For this reason τ is called the time to the initial collision of a packing sequence.

For certain choices of S, μ, and δ, τ is a random variable for which some results are known. For example, several varieties of what are commonly called birthday problems can be treated by suitably defining S, μ, and δ. In Sections 1.3, 1.4, 1.7,and 1.8 which follow, some known results concerning τ will be reviewed and some extensions and new results will be obtained, for some particular metric spaces. Simple applications will be mentioned, and some examples will be presented. Sections 1.5 and 1.6 will examine some related problems for which not all of the criteria needed for a packing sequence as defined here are met. However before proceeding with any specific packing sequence, it is convenient to derive first some results which are true for a large class of choices of $\langle S, \mu\rangle$ and δ.

1.2. Sequences of Equivalent Points

A packing sequence will be said to consist of equivalent points if the probability that a randomly selected point collides with any given element of the space S is equal to the probability that two arbitrary points collide. That is, the sequence has equivalent points if for each i and for every $s \in S$

$$P\{C_i \wedge s\} = P\{\mu(C_i, s) < \delta\} = p.$$

It will be seen in the next section that the packing sequence associated with the classical birthday problem consists of equivalent points. So do the remaining packing sequences of Section 1.3, and also those discussed in Sections 1.4 and 1.7. The first

packing sequence discussed in Section 1.8 consists of equivalent points; however, it will be shown that the alternative packing sequence for ternary n-tuples found in that section does not.

As a simple example of a packing sequence which does not consist of equivalent points, consider the one having $S = \{1,2,3,4,5,6,7\}$, $\mu(x,y) = |x-y|$, and $\delta = 3$. Here $P\{C_i \wedge 1\} = P\{C_i \wedge 7\} = \frac{3}{7}$, $P\{C_i \wedge 2\} = P\{C_i \wedge 6\} = \frac{4}{6}$, and $P\{C_i \wedge 3\} = P\{C_i \wedge 4\} = P\{C_i \wedge 5\} = \frac{5}{7}$. Since

$$p = \frac{2}{7}\left(\frac{3}{7}\right) + \frac{2}{7}\left(\frac{4}{7}\right) + \frac{3}{7}\left(\frac{5}{7}\right) = \frac{29}{49}$$

it is clear that the sequence does not possess the equivalent points property that $P\{C_i \wedge s\} = p$ for every $s \in S$. In both this example and the non-equivalent points case of Section 1.8, it can be noted that the metric μ creates a boundary effect on the space S. If the metric above is changed to

$$\mu(x,y) = \begin{cases} |x-y| & \text{if } |x-y| \leq 3 \\ 7 - |x-y| & \text{otherwise} \end{cases}$$

in order to impose a torus quality on the points of S, then the packing sequence now consists of equivalent points. This is because for each $s \in S$

$$P\{C_i \wedge s\} = p = \frac{5}{7}.$$

The results presented in this section hold for any $\delta > 0$ and metric space $\langle S, \mu \rangle$ which provide a packing sequence of equivalent points. The corresponding uniform probability measure P will always be as specified in Section 1.1.

A lower bound for $P\{\tau > k\}$ is easy to establish.

Proposition 1.1. For $k \geq 2$,

$$P\{\tau > k\} \geq \begin{cases} \prod_{j=1}^{k-1}(1 - jp) & \text{if } p < (k-1)^{-1} \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Using Boole's inequality

$$\begin{aligned}
P\{\tau = k \mid \tau > k-1\} &= P\{C_1 \wedge C_k \cup C_2 \wedge C_k \cup \cdots \cup C_{k-1} \wedge C_k \mid \tau > k-1\} \\
&\leq \sum_{j=1}^{k-1} P\{C_j \wedge C_k \mid \tau > k-1\} \\
&= \sum_{j=1}^{k-1} P\{C_j \wedge C_k\} \\
&= (k-1)p.
\end{aligned}$$

Thus

$$\begin{aligned}
P\{\tau > k\} &= P\{\tau > k-1\} - P\{\tau = k\} \\
&= (1 - P\{\tau = k \mid \tau > k-1\})P\{\tau > k-1\} \\
&\geq (1-(k-1)p)P\{\tau > k-1\}.
\end{aligned}$$

For $p \leq (k-1)^{-1}$, simple induction yields

$$P\{\tau > k\} \geq (1-(k-1)p)(1-(k-2)p)\cdots(1-p)P\{\tau > 1\}.$$

Noting that $P\{\tau > 1\} = 1$ completes the proof. ∎

It can be seen by examining (1.11) of the next section that this lower bound is sharp for the sequence of the simple birthday problem. Thus, on the average, the waiting time for a coincidence will tend to be at least as long for any other packing sequence consisting of equivalent points and having the same value for p (equal to n^{-1}) as it is for the corresponding simple birthday sequence.

Since $\prod_{j=1}^{k-1}(1-jp)$ becomes cumbersome to compute for large k, it is desirable to establish a more convenient lower bound. Lemma 1.1 gives a lower bound which can be easily used to establish asymptotic results concerning the distribution of τ.

Lemma 1.1. For any real numbers n and α satisfying $n \geq 3$ and $\alpha \leq \frac{2}{3}$, and any integer k such that $2 \leq k < n$,

$$\prod_{j=1}^{k-1}\left(1 - \frac{j}{n}\right) > \begin{cases} e^{-k^2/2n}(1-kn^{-\alpha}) & \text{if } k < n^{\alpha} \\ 0 & \text{otherwise} \end{cases}$$

Proof. The case for $k \geq n^\alpha$ is trivial. For $k < n^\alpha$ observe that

$$\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right)=\frac{\Gamma(n)}{n^{k-1}(n-k)\Gamma(n-k)}.$$

Judicious application of the double inequality

$$\sqrt{2\pi}x^{x-1/2}e^{-x} < \Gamma(x) < \sqrt{2\pi}x^{x-1/2}e^{-x+\frac{1}{12x}} \tag{1.1}$$

to the gamma functions above yields

$$\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right) > \left(1-\frac{k}{n}\right)^{-n+k-1/2}\exp\left(-k-\frac{1}{12(n-k)}\right).$$

Hence it suffices to show

$$\left(1-\frac{k}{n}\right)^{-n+k-1/2}\left(1-\frac{k}{n^\alpha}\right)^{-1}\exp\left(\frac{k^2}{2n}-k-\frac{1}{12(n-k)}\right)\geq 1.$$

or equivalently

$$\left(n-k+\frac{1}{2}\right)\log\left(1-\frac{k}{n}\right)^{-1}+\log\left(1-\frac{k}{n^\alpha}\right)^{-1}+\frac{k^2}{2n}-k-\frac{1}{12(n-k)}\geq 0. \tag{1.2}$$

Since the expansion

$$\log\left(\frac{1}{1-t}\right)=t+\frac{1}{2}t^2+\frac{1}{3}t^3+\cdots\ (|t|<1)$$

yields the inequality

$$\log(1-t)^{-1}\geq\sum_{m=1}^{M}\frac{1}{m}t^m \quad (0<t<1,\ M\geq 1), \tag{1.3}$$

it follows that the left side of (1.2) exceeds

$$\begin{aligned}&(n-k)\left(\frac{k}{n}+\frac{1}{2}\left(\frac{k}{n}\right)^2\right)+\frac{k}{n^\alpha}+\frac{k^2}{2n}-k-\frac{1}{12(n-k)}\\ &=\frac{k}{n^\alpha}-\frac{k^3}{2n^2}-\frac{1}{12(n-k)}.\end{aligned} \tag{1.4}$$

Thus it suffices to show that (1.4) is greater than zero, or equivalently that

$$12(n-k)\frac{k}{n^\alpha}\left(1-\frac{k^2}{2n^{2-\alpha}}\right)>1. \tag{1.5}$$

But this is indeed the case since the left side of (1.5) exceeds

$$\begin{aligned}12(n-n^{\alpha})\frac{2}{n^{\alpha}}\left(1-\frac{n^{3\alpha-2}}{2}\right)\\&= 24(n^{1-\alpha}-1)\left(1-\frac{n^{3\alpha-2}}{2}\right)\\&> 24(n^{1/3}-1)\left(1-\frac{n^{0}}{2}\right)\\&\geq 12(3^{1/3}-1)\\&> 1. \qquad \blacksquare\end{aligned}$$

The corollary below follows immediately from Proposition 1.1 and Lemma 1.1.

Corollary. For α real and k an integer, if p, α and k satisfy $p \leq \frac{1}{3}$, $\alpha \leq \frac{2}{3}$ and $k \geq 2$, then

$$P\{\tau > k\}\begin{cases} > e^{-\frac{k^2}{2}p}(1-kp^{\alpha}) & \text{if } k \leq p^{-\alpha} \\ \geq 0 & \text{otherwise.}\end{cases}$$

An alternative lower bound for $\prod_{j=1}^{k-1}(1-jp)$ can be obtained by a different technique. Although this lower bound is tighter than the one found above, note that the nontrivial portion still requires that k does not exceed $n^{2/3}$.

Lemma 1.2. For any $n \geq 3$ and any integer $k \geq 2$,

$$\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right)\begin{cases} > e^{-\frac{k^2}{2n}}\left(1-\frac{k^3}{3n^2}\right) & \text{if } k \leq n^{2/3} \\ \geq 0 & \text{otherwise.}\end{cases}$$

Proof. Since

$$\log(1-t) > -t-t^2$$

for $0 < t \leq \frac{3}{5}$, and since $\frac{j}{n} \leq \frac{3}{5}$ for $1 \leq j \leq k-1$ whenever the conditions of the hypothesis

are met and $k \le n^{2/3}$, it follows that if $k \le n^{2/3}$ then

$$\begin{aligned}
\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right) &= \exp\left\{\sum_{j=1}^{k-1}\log\left(1-\frac{j}{n}\right)\right\} \\
&> \exp\left\{\sum_{j=1}^{k-1}\left(-\frac{j}{n}-\left(\frac{j}{n}\right)^2\right)\right\} \\
&= \exp\left\{-\frac{(k-1)k}{2n}-\frac{(k-1)k(2k-1)}{6n^2}\right\} \\
&> \exp\left\{-\frac{k^2}{2n}-\frac{k^3}{3n^2}\right\}.
\end{aligned}$$

Then since $\epsilon^{-t} > 1-t$ for all $t \le 3$ and since $\frac{k^3}{3n^2} \le \frac{1}{3}$, it follows from above that if $k \le n^{2/3}$ then

$$\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right) > \epsilon^{-\frac{k^2}{2n}}\left(1-\frac{k^3}{3n^2}\right). \qquad \blacksquare$$

Corollary. For any $p \le \frac{1}{3}$ and any integer $k \ge 2$,

$$P\{\tau > k\}\begin{cases} > \epsilon^{-\frac{k^2}{2}p}\left(1-\frac{1}{3}k^3p^2\right) & \text{if } k \le p^{-2/3} \\ \ge 0 & \text{otherwise.}\end{cases}$$

It is also possible to establish an upper bound for $P\{\tau > k\}$.

Proposition 1.2. Let M be the largest integer m satisfying

$$\frac{(2m^2-5m+1)}{4}p + \frac{(m-1)^5}{24}p^2 \le 1.$$

Then

$$P\{\tau > k\} < \begin{cases} e^{-\frac{(k-1)^2}{2}p} & \text{for } 2 \le k < M \\ e^{-\frac{(M-1)^2}{2}p} & \text{otherwise.}\end{cases}$$

Proof. Since $P\{C_r \wedge C_s, C_t \wedge C_u\} = p^2$ if the set $\{r,s\}$ is not identical to the set $\{t,u\}$, it follows by a well-known inclusion-exclusion argument that for $k \ge 3$,

$$\begin{aligned}
P\{\tau > k\} &= P\left\{\bigcap_{1\le i<j\le k} C_i \vee C_j\right\} \\
&\le 1 - \sum_{r<s} P\{C_r \wedge C_s\} + \sum_{\substack{r<s,t<u,s\le u \\ \{r,s\}\ne\{t,u\}}} P\{C_r \wedge C_s, C_t \wedge C_u\}
\end{aligned}$$

$$= 1 - \binom{k}{2} p + \binom{\binom{k}{2}}{2} p^2. \tag{1.6}$$

Now truncating the Taylor expansion of $e^{-\frac{(k-1)^2}{2}p}$ yields

$$e^{-\frac{(k-1)^2}{2}p} > 1 - \frac{(k-1)^2}{2}p + \frac{(k-1)^4}{8}p^2 - \frac{(k-1)^6}{48}p^3 \quad \left(k \le 1 + \sqrt{\frac{10}{p}}\right). \tag{1.7}$$

The condition that (1.6) is less than or equal to the right side of (1.7) simplifies to

$$\frac{(2k^2 - 5k + 1)}{4}p + \frac{(k-1)^5}{24}p^2 \le 1. \tag{1.8}$$

Hence it follows that

$$P\{\tau > k\} < e^{-\frac{(k-1)^5}{2}p}$$

whenever (1.8) is true and $3 \le k < 1 + \sqrt{\frac{10}{p}}$. Over $k \ge 2$, the left side of (1.8) increases with k and is not bounded from above, so that the existence of a unique M is guaranteed. Furthermore, whatever p is, $2 \le M < 1 + \sqrt{\frac{10}{p}}$. The proposed upper bound has now been established for $3 \le k \le M$. For $k > M$ it is trivial that

$$P\{\tau > k\} \le P\{\tau > M\} < e^{-\frac{(M-1)^2}{2}p}.$$

For $k = 2$ note that

$$P\{\tau > 2\} = 1 - p < e^{-p/2}. \qquad \blacksquare$$

Corollary. Let

$$M(p) = \begin{cases} \lfloor p^{-1/2} \rfloor & \text{if } p \ge \frac{1}{144} \\ \left\lfloor \left(\frac{12}{p^2}\right)^{1/5} \right\rfloor & \text{otherwise.} \end{cases}$$

Then

$$P\{\tau > k\} < \begin{cases} e^{-\frac{(k-1)^2}{2}p} & \text{for } 2 \le k \le M(p) \\ e^{-\frac{M^2(p)}{2}p} & \text{otherwise.} \end{cases}$$

Proof. Note that $M(p) = \min\{[\![p^{-1/2}]\!], [\![\left(\frac{12}{p^2}\right)^{1/5}]\!]\}$. Hence

$$\begin{aligned}
&\frac{(2m^2 - 5m + 1)}{4}p + \frac{(m-1)^5}{24}p^2\Big|_{m=M(p)+1} \\
&\le \frac{(m-1)^2}{2}p + \frac{(m-1)^5}{24}p^2\Big|_{m=M(p)+1} \\
&\le \frac{[\![\frac{1}{\sqrt{p}}]\!]^2}{2}p + \frac{[\![(\frac{12}{p^2})^{1/5}]\!]^5}{24}p^2 \\
&\le 1,
\end{aligned}$$

and so $M(p) + 1$ does not exceed the value M specified in the hypothesis of Proposition 1.2. Thus it follows from the proposition, and the fact that $P\{\tau > k\}$ is nonincreasing with k, that the proposed upper bound is valid. ∎

Proposition 1.3. For fixed $t > 0$,

$$P\left\{\frac{\tau}{\sqrt{2/p}} > t\right\} \sim e^{-t^2} \quad \text{as} \quad p \downarrow 0.$$

Proof. For p sufficiently small, the preceding corollary may be used to attain the following upper bound.

$$\begin{aligned}
P\left\{\frac{\tau}{\sqrt{2/p}} > t\right\} &= P\{\tau > [\![t\sqrt{2/p}]\!]\} \\
&< \exp\left\{-\frac{1}{2}([\![t\sqrt{2/p}]\!] - 1)^2 p\right\} \\
&< \exp\{-t^2 - 4t\sqrt{p/2}\}.
\end{aligned}$$

Similarly, by the corollary following Lemma 1.1 (with $\alpha = \frac{2}{3}$),

$$P\left\{\frac{\tau}{\sqrt{2/p}} > t\right\} > \exp\{-t^2 - \sqrt{2}t\, p^{1/6}\}.$$

These two bounds, plus the sandwiching theorem for limits, yield

$$\lim_{p \to 0^+} \frac{P\{\tau\sqrt{p/2} > t\}}{e^{-t^2}} = 1$$

as required. ∎

It should be noted that for the special case of the simple birthday problem the result given in the previous proposition is known to Diaconis [9] and probably others who have studied the problem.

The result stated in the next proposition is of a different form than the more common types of collision problem results, as many previous results treat $P\{\tau > k\}$ for either k fixed or k being constrained to be of the order $O(p^{-1/2})$. The restriction that α be less than $\frac{2}{5}$ arises from a similar restriction in Proposition 1.2. In Sections 1.3 and 1.4, where the distribution of τ is known exactly, the following results will be strengthened to hold for all $0 < \alpha < \frac{2}{3}$. For the special case of $\alpha = \frac{1}{2}$ this result may be obtained for numerous settings from a Poisson limit of Silverman and Brown [52] Their results, which arise from a method involving the U-statistics of Hoeffding [21], will be discussed further in the next section.

Proposition 1.4. For $0 < \alpha \leq \frac{2}{5}$,

$$P\{\tau > [\![p^{-\alpha}]\!]\} \sim \exp(-\frac{1}{2}p^{1-2\alpha}) \quad \text{as} \quad p \downarrow 0.$$

Proof. Note that

$$\begin{aligned}\left(\frac{12}{p^2}\right)^{1/5} - p^{-\alpha} &\geq \frac{(12^{1/5}) - 1}{p^{2/5}} \\ &> 0\end{aligned}$$

so that $[\![p^{-\alpha}]\!]$ must be less than or equal to $[\![(\frac{12}{p^2})^{1/5}]\!]$. Hence, by the corollary to Proposition 1.2,

$$\begin{aligned}\limsup_{p\to 0^+} \frac{P\{\tau > [\![p^{-\alpha}]\!]\}}{\exp(-\frac{1}{2}p^{1-2\alpha})} &\leq \limsup_{p\to 0^+} \exp\left[\frac{1}{2}p^{1-2\alpha} - \frac{([\![p^{-\alpha}]\!]-1)^2}{2}p\right] \\ &\leq \limsup_{p\to 0^+} \exp\left[\frac{1}{2}p^{1-2\alpha} - \frac{(p^{-\alpha}-2)^2}{2}p\right] \\ &= \limsup_{p\to 0^+} \exp(2p^{1-\alpha} - 2p) \\ &= 1.\end{aligned}$$

Also, by the corollary following Lemma 1.1,

$$\liminf_{p\to 0^+} \frac{P\{\tau > [p^{-\alpha}]\}}{\exp(-\frac{1}{2}p^{1-2\alpha})} \geq \liminf_{p\to 0^+}(1-[p^{-\alpha}]p^{1/2})\exp\left(\frac{1}{2}p^{1-2\alpha} - \frac{[p^{-\alpha}]^2}{2}p\right)$$
$$\geq \liminf_{p\to 0^+}(1-p^{1/2-\alpha})\exp(\frac{1}{2}p^{1-2\alpha} - \frac{1}{2}p^{1-2\alpha})$$
$$= 1,$$

so that it may now be concluded that

$$\lim_{p\to 0^+} \frac{P\{\tau > [p^{-\alpha}]\}}{\exp(-\frac{1}{2}p^{1-2\alpha})} = 1. \quad \blacksquare$$

Next a lower bound for $E[\tau]$ will be given. A similar upper bound for $E[\tau]$ cannot be produced (by the author) in this general setting since a sufficiently tight upper bound for $P\{\tau > k\}$ has not been determined over a large enough range of k. In subsequent sections, where specific metric spaces are considered, nice upper bounds for $E[\tau]$ are found in those cases where the distribution of τ is known precisely.

Theorem 1.1. Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$. Then for $p \leq \frac{2}{3} - \alpha$

$$E[\tau] > \sqrt{\frac{\pi}{2p}} - p^{\alpha-1}.$$

T[illegible] [illegible]ower bound is asymptotically equivalent to $\sqrt{\frac{\pi}{2p}}$ as $p \downarrow 0$.

Proof. Since τ is a nonnegative, integral-valued random variable, it follows from a well-known argument that

$$E[\tau] = \sum_{k=0}^{\infty} P\{\tau > k\}.$$

However, since $\tau \geq 2$ with probability 1, it is also true that

$$E[\tau] = 2 + \sum_{k=2}^{\infty} P\{\tau > k\},$$

and so by the corollary following Lemma 1.1

$$E[\tau] > 2 + \sum_{k=2}^{[p^{-\alpha}]} e^{-\frac{k^2}{2}p}(1 - kp^{\alpha}).$$

Since $e^{-\frac{x^2}{2}p}(1 - xp^{\alpha})$ is a positive, decreasing function of x on $(2, p^{-\alpha})$,

$$\begin{aligned} E[\tau] &> 2 + \sum_{k=2}^{[p^{-\alpha}]-1} \int_k^{k+1} e^{-\frac{x^2}{2}p}(1 - xp^{\alpha})dx + \int_{[p^{-\alpha}]}^{p^{-\alpha}} e^{-\frac{x^2}{2}p}(1 - xp^{\alpha})dx \\ &= 2 + \int_2^{p^{-\alpha}} e^{-\frac{x^2}{2}p}(1 - p^{\alpha})dx. \end{aligned}$$

Since $e^{-\frac{x^2}{2}p}(1 - xp^{\alpha})$ is less than 1 on $(0, 1)$ and is negative on $(p^{-\alpha}, \infty)$, it further follows that

$$\begin{aligned} E[\tau] &> \int_0^{\infty} e^{-\frac{x^2}{2}p}(1 - xp^{\alpha})dx \\ &= \sqrt{\frac{\pi}{2p}} - \frac{1}{p^{1-\alpha}} \end{aligned}$$

as claimed. Also note that

$$\lim_{p \to 0^+} \frac{\sqrt{\frac{\pi}{2p}} - p^{\alpha-1}}{\sqrt{\frac{\pi}{2p}}} = \lim_{p \to 0^+} \left(1 - p^{\alpha - 1/2}\sqrt{\frac{2}{\pi}}\right) = 1. \quad \blacksquare$$

1.3. The Birthday Surprise Revisited

Let $S = \{1, 2, \cdots, n\}$ and let

$$\mu(x, y) = \begin{cases} |x - y| & \text{if } |x - y| \le \frac{n}{2} \\ n - |x - y| & \text{otherwise.} \end{cases}$$

Recalling that δ represents the minimum separation distance for disjoint points, consider first the simple case where δ is taken to be 1. Then two points $C_i \in S$ and $C_j \in S$ will collide if and only if $\mu(C_i, C_j) = 0$, a condition requiring that C_i and C_j be the same element from S.

The sequential selection of $C_1, C_2, \cdots$ from S may be thought of as dropping balls labeled $C_1, C_2, \cdots$ into cells labeled $1, 2, \cdots, n$. For each drop, the probability of the ball landing in cell j is just $\frac{1}{n}$ $(j = 1, 2, \cdots, n)$. Also the probability that two arbitrary points C_i and C_j collide is $\frac{1}{n}$, since whatever cell the ball C_i falls into there exists a 1 in n chance that C_j will occupy the same cell. Clearly, the packing sequence consists of equivalent points.

The initial collision in the sequence will occur the first time a dropped ball lands in a cell which is already occupied by a previously allocated ball, i.e. τ counts the number of drops required to produce the first double occupancy. Problems concerning the distribution of τ are generally referred to as birthday problems. A common example, sometimes called the birthday surprise, is to determine the smallest number of people needed in order for there to be at least an even chance that some pair of them will share the same birthday, i.e. for $n = 365$ find the smallest value of k such that $P\{\tau > k\} \leq \frac{1}{2}$.

Some basic aspects of birthday problems are discussed in [14], and numerous authors have investigated extensions and generalizations of the simple versions of the problem. McKinney [37] calculates the probability that at least r out of n randomly selected people have the same birthday, and Klamkin and Newman [31] determine, asymptotically, the expected number of people needed in order for r of them to have the same birthday. If there are n days in the year, then the expected number needed is asymptotically equivalent to

$$(r!)^{1/r}\Gamma\left(1+\frac{1}{r}\right) n^{(1-\frac{1}{r})}$$

as n tends to infinity. For the case of $r = 2$, their theorem yields the result

$$E[\tau] \sim \sqrt{\frac{n\pi}{2}} \quad \text{as} \quad n \to \infty$$

where, as before, τ is the time of the first match. Blaum et al. [6] present more accurate asymptotic estimates than those given in [31], and they also consider the number of people needed in order to have k different birthdays occur at least r times each. Several other variations are to be discussed in a forthcoming book by Diaconis and Mosteller, among them are problems in which birthdays do not occur with equal likelihood, multivariate versions, and "near" coincidences. Among their findings in each of the settings investigated is an approximation for the number of people needed in order for there to be an even chance for a match.

Now consider the case where δ equals 2. If the n cells are arranged in a circle, and balls are dropped into them at random, then τ counts the number of balls required

to obtain the first occurrence of either any cell being doubly occupied or of any pair of adjacent cells being singly occupied. Clearly

$$p = P\{C_i \wedge C_j\} = \frac{3}{n} \quad (i \neq j),$$

since whatever cell C_i occupies there will be a collision if C_j lands in the same cell or either of the two cells adjacent to it.

Another way to view the situation is as follows. Consider a circle of circumference n which is divided into n segments, each segment being of arc length 1. The segments are labeled in order $1, 2, \cdots, n$ so that the nth segment is adjacent to the 1st segment. If $C_i = j$ $(\neq n)$ then place an arc of length 2 on the circle so that it covers exactly the jth and $(j+1)$th segments. If $C_i = n$ then let an arc cover the nth and 1st segments. A collision will occur whenever any portion of two arcs are overlapping.

This circular representation is also convenient if $\delta \in \{3, 4, \cdots\}$. At each step, if $C_i = j$ then cover the segments labeled $j, j \oplus 1, \cdots, j \oplus (\delta - 1)$ with an arc of length δ, where $\oplus$ stands for addition modulo n. The initial collision occurs the first time any portion of the circle becomes twice covered. Thus τ equals k if the first $k-1$ arcs are pairwise disjoint and the kth arc overlaps at least one of the first $k-1$ arcs. Note that $\mu(C_i, C_j) = 0$ if $C_i = C_j$. Otherwise $\mu(C_i, C_j)$ is just one greater than the number of segments which lie between the C_ith segment and the C_jth segment, where the shortest possible portion of the circle connecting the two segments is considered.

For convenience, in this metric space the stopping time τ will be written $\tau_{\delta,n}$, where δ identifies the minimum separation for disjoint points and n is the number of elements in S. For example, $P\{\tau_{7,365} > k\}$ is just the probability, under reasonable simplifying assumptions, that in a group of k people sampled at random, no pair of them have birthdays less than a week apart. This type of birthday problem seems to have been investigated first by Abramson and Moser [1], and subsequent work on extensions has been done by Sevast'yanov [48] and Diaconis [9] among others.

For another example, consider the packing problem discussed in [43] and [36]. There

a line of integral length n is filled sequentially at random with nonoverlapping intervals of integral length a, their end points having integer coordinates. If a selected interval were to overlap one which has already been packed, then the interval is rejected and another one is selected at random. The process continues until it is no longer possible to fairly pack another interval on the line. Disregarding end effects, the event $\{\tau_{a,n} > k\}$ corresponds to having none of the first k intervals rejected.

For $c \in \{1, 2, \cdots\}$, $\tau_{c,n}$ is a random variable having $\{2, 3, \cdots, [\![\frac{n}{c}]\!] + 1\}$ as its set of possible outcomes. $P\{\tau_{c,n} > k\} = 1$ for all integers $k < 2$ and $P\{\tau_{c,n} > k\} = 0$ for all integers $k > [\![\frac{n}{c}]\!]$. For $2 \le k \le [\![\frac{n}{c}]\!]$, Abramson and Moser [1] proved that

$$
\begin{aligned}
P\{\tau_{c,n} > k\} &= \frac{(k-1)!}{n^{k-1}} \binom{n - k(c-1) - 1}{k-1} \\
&= \frac{(n - k(c-1) - 1)(n - k(c-1) - 2)\cdots(n - k(c-1) - (k-1))}{n^{k-1}}.
\end{aligned}
\tag{1.9}
$$

An equivalent form is

$$
P\{\tau_{c,n} > k\} = \prod_{j=1}^{k-1} \left[1 - \left(\frac{k(c-1) + j}{n}\right)\right]. \tag{1.10}
$$

Setting c equal to 1 yields

$$
P\{\tau_{1,n} > k\} = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) \quad (k = 2, \cdots, n), \tag{1.11}
$$

a well-known expression for the probabilities of the basic birthday problem.

The major advancement contained in this section is given by Proposition 1.10, which states that for $0 < \alpha < \frac{2}{3}$

$$
P\left\{\tau_{c,n} > \left[\!\!\left[\left(\frac{n}{2c-1}\right)^{\alpha}\right]\!\!\right]\right\} \sim \exp\left[-\frac{1}{2}\left(\frac{n}{2c-1}\right)^{2\alpha - 1}\right] \quad \text{as} \quad n \to \infty.
$$

Note that this result implies that, for large n, only for values of k near $\sqrt{n}$ will $P\{\tau_{c,n} > k\}$ be something other than almost zero or almost one, a fact which is known by Diaconis [9] and perhaps others who have investigated the birthday problem. However, note that the result also provides a way of estimating what the precise value of $P\{\tau_{c,n} > k\}$ is when it is close to 0 or 1, provided that α is not too large.

Letting the qth quantile of $\tau_{c,n}$, denoted by ξ_q, be defined as the smallest integer ξ satisfying $P\{\tau \leq \xi\} \geq q$, the above proposition suggests that ξ_q should be reasonably approximated by

$$\hat{\xi}_q = \left\lceil \sqrt{(\frac{2n}{2c-1})\log(\frac{1}{1-q})} \right\rceil$$

if $q < 1 - \exp\left[-\frac{1}{2}\left(\frac{n}{2c-1}\right)^{1/3}\right]$. It turns out, however, that the approximation is rather good for even larger q. This is demonstrated by the following example. Let $c = 2$ and $n = 3000$. Then $1 - \exp\left[-\frac{1}{2}\left(\frac{n}{2c-1}\right)^{1/3}\right] = 1 - e^{-5} \doteq .993$. A comparison of ξ_q and $\hat{\xi}_q$ for various values of q is presented below.

q	ξ_q	$\hat{\xi}_q$
0.5	38	38
0.9	68	68
0.99	96	96
0.993	99	100
0.995	103	103
0.999	117	118
0.9995	122	124
0.9999	134	136
$1 - 10^{-5}$	150	152
$1 - 10^{-7}$	176	180
$1 - 10^{-9}$	199	204

It is interesting to note that although the difference between 122 and 124 seems slight, the difference between $P\{\tau > 122\} = 4.88 \times 10^{-4}$ and $P\{\tau > 124\} = 3.77 \times 10^{-4}$ is more pronounced. Similarly, the difference between 199 and 204 does not appear to be nearly as drastic as the difference between $P\{\tau > 199\} = 9.3 \times 10^{-10}$ and $P\{\tau > 204\} = 3.1 \times 10^{-10}$.

For the special case of $\alpha = \frac{1}{2}$, the result of Proposition 1.10 has been obtained by several investigators. These include Sevast'yanov [48], Silverman and Brown [52], Diaconis [9], and Stein [56]. In fact, for this special case it is possible to state more powerful results

than that which is furnished by Proposition 1.10. These more powerful results are generally known as Poisson approximations.

Consider the basic birthday problem where there are k people, n days in a year, and it is considered that a match occurs whenever a pair of people have a common birthday. For this setting the Poisson approximation loosely says that if $\binom{k}{2}\frac{1}{n}$ is approximately equal to some fixed value λ and n is sufficiently large, then

$$P\{\#\text{ matches } = j\} \cong \frac{e^{-\lambda}\lambda^j}{j!}.$$

It is not even necessary to require that all birthdays occur with equal likelihood, although some restrictions on the probabilities are needed. Note that since k is required to be of the order $O(n^{1/2})$, the $j = 0$ version of the Poisson approximation is indeed similar to the $\alpha = \frac{1}{2}$ case of Proposition 1.10. Similar Poisson approximations can be obtained in collision settings other than the one discussed here.

Now methods similar to those of the previous section will be used to develop bounds for $P\{\tau_{c,n} > k\}$. These bounds will in turn be used to establish limit results for the distribution of $\tau_{c,n}$, including Proposition 1.10.

It follows from (1.10) that

$$\log P\{\tau_{c,n} > k\} = \sum_{j=1}^{k-1} \log\left[1 - \left(\frac{k(c-1)+j}{n}\right)\right]. \tag{1.12}$$

The inequality

$$\log(1-x) < -x \quad (0 < x < 1)$$

may be applied to each of the terms on the right hand side of (1.12), resulting in

$$\begin{aligned}\log P\{\tau_{c,n} > k\} &< \sum_{j=1}^{k-1}\left[-\left(\frac{k(c-1)+j}{n}\right)\right] \\ &= -\frac{(k-1)k(2c-1)}{2n}.\end{aligned}$$

Thus the following proposition is immediately established.

Proposition 1.5.

$$P\{\tau_{c,n} > k\} < e^{-\frac{(k-1)k(2c-1)}{2n}} \qquad \left(k = 2, \cdots, \left[\left[\frac{n}{c}\right]\right]\right).$$

Noting that $p = \frac{(2c-1)}{n}$, the next proposition is just a special case of the corollary following Lemma 1.1.

Proposition 1.6. For α real and k an integer, if $n \geq 3(2c-1)$, $\alpha \leq \frac{2}{3}$ and $k \geq 2$, then

$$P\{\tau_{c,n} > k\} \begin{cases} > \left[1 - k\left(\frac{2c-1}{n}\right)^{\alpha}\right] \exp\left(-k^2 \frac{(2c-1)}{2n}\right) & \text{if } k < \left(\frac{n}{2c-1}\right)^{\alpha} \\ \geq 0 & \text{otherwise.} \end{cases}$$

An alternative lower bound is provided by the following proposition.

Proposition 1.7. Let c, n, and k be integers such that $2 \leq c \leq 1728$, $n \geq c^3$, and $k \geq 2$. Let

$$M(c,n) = \begin{cases} \left(\frac{n}{2}\right)^{2/3} & \text{if } c < 1 + \left(\frac{n}{2}\right)^{1/3} \\ \frac{n}{2(c-1)} & \text{otherwise.} \end{cases}$$

Then

$$P\{\tau_{c,n} > k\} \begin{cases} > \left(1 - \frac{k}{(n-k(c-1))^{2/3}}\right) \exp\left(-\frac{k^2(2c-1)}{2(n-k(c-1))}\right) & \text{if } k < M(c,n) \\ \geq 0 & \text{otherwise.} \end{cases}$$

Proof. Consider $k < M(c,n)$ since the other case is trivial.

For convenience let γ denote $(c-1)$. Note that

$$M(c,n) = \min\left\{\left(\frac{n}{2}\right)^{2/3}, \frac{n}{2\gamma}\right\}$$

so that

$$M^{3/2} + \gamma M \leq n,$$

or equivalently

$$M \leq (n - \gamma M)^{2/3}.$$

Therefore, for $k < M(c,n)$

$$k < (n - \gamma k)^{2/3}.$$

It follows from (1.9) and (1.1) that

$$\begin{aligned}
P\{\tau_{c,n} > k\} &= \frac{\Gamma(n-k\gamma)}{n^{k-1}(n-k\gamma-k)\Gamma(n-k\gamma-k)} \\
&> \frac{(n-k\gamma)^{n-k\gamma-1/2}\exp(-n+k\gamma)}{n^{k-1}(n-k\gamma-k)^{n-k\gamma-k+1/2}\exp(-n+k\gamma+k+\frac{1}{12(n-k\gamma-k)})} \\
&= \left[\left(1-\frac{1}{\left(\frac{n}{k\gamma}\right)}\right)^{\left(\frac{n}{k\gamma}-1\right)}\right]^{\frac{(k-1)k\gamma}{(n-k\gamma)}} \left(\frac{1}{1-\left(\frac{k}{nk\gamma}\right)}\right)^{n-k\gamma-k+1/2} \\
&\quad \times \exp\left(-k-\frac{1}{12(n-k\gamma-k)}\right).
\end{aligned}$$

The inequality

$$\left(1-\frac{1}{x}\right)^{x-1} > e^{-1} \quad (x>1)$$

applied to the expression above yields

$$P\{\tau_{c,n} > k\} > \left[1-\left(\frac{k}{n-k\gamma}\right)\right]^{-n+k\gamma+k-1/2} \exp\left(-\frac{(k-1)k\gamma}{(n-k\gamma)}-k-\frac{1}{12(n-k\gamma-k)}\right).$$

It now suffices to show that if $k < (n-\gamma k)^{2/3}$ then

$$\begin{aligned}
&\left[1-\left(\frac{k}{n-k\gamma}\right)\right]^{-n+k\gamma+k-1/2}\left[1-\frac{k}{(n-k\gamma)^{2/3}}\right]^{-1} \\
&\quad \times \exp\left(\frac{k^2(2\gamma+1)}{2(n-k\gamma)}-\frac{(k-1)k\gamma}{(n-k\gamma)}-k-\frac{1}{12(n-k\gamma-k)}\right).
\end{aligned}$$

exceeds 1, or equivalently that

$$\begin{aligned}
&(n-k\gamma-k+\frac{1}{2})\log\left[1-\frac{k}{(n-k\gamma)}\right]^{-1}+\log\left[1-\frac{k}{(n-k\gamma)^{2/3}}\right] \\
&\quad +\frac{k^2(2\gamma+1)}{(n-k\gamma)}-\frac{(k-1)k\gamma}{(n-k\gamma)}-k-\frac{1}{12(n-k\gamma-k)}
\end{aligned}$$

is greater than 0. By (1.3) the above expression exceeds

$$\begin{aligned}
&(n-k\gamma-k)\left[\frac{k}{(n-k\gamma)}+\frac{k^2}{2(n-k\gamma)^2}\right]+\frac{k}{(n-k\gamma)^{2/3}}+\frac{k^2(2\gamma+1)}{2(n-k\gamma)}-\frac{(k-1)k\gamma}{(n-k\gamma)} \\
&\quad -k-\frac{1}{12(n-k\gamma-k)} \\
&= \frac{k}{2(n-k\gamma)^2}[2\gamma(n-k\gamma)-k^2+2(n-k\gamma)^{4/3}]-\frac{1}{12(n-k\gamma-k)} \\
&\geq \frac{k}{2(n-k\gamma)^2}[2(n-k\gamma)^{4/3}-k^2]-\frac{1}{12(n-k\gamma-k)}.
\end{aligned}$$

Therefore, it suffices to show

$$\frac{k}{2(n-k\gamma)^2}[2(n-k\gamma)^{\frac{4}{3}}-k^2] > \frac{1}{12(n-k\gamma-k)},$$

or equivalently

$$\frac{6(n-k\gamma-k)k}{(n-k\gamma)^2}[2(n-k\gamma)^{4/3}-k^2] > 1. \tag{1.13}$$

Now for all allowable values of n, k and γ,

$$\begin{aligned}\frac{6(n-k\gamma-k)k}{(n-k\gamma)^2} &= 6\left[1-\frac{k}{(n-k\gamma)}\right]\left[\frac{k}{n-k\gamma}\right]\\ &\geq 6\left(1-\frac{2}{n}\right)\left(\frac{2}{n}\right)\\ &\geq \frac{12}{n}\left(1-\frac{2}{c^3}\right)\end{aligned}$$

and

$$\begin{aligned}[2(n-k\gamma)^{4/3}-k^2] &\geq (n-k\gamma)^{4/3}\\ &> (n-n^{2/3}\gamma)^{4/3}\\ &= n^{4/3}(1-\gamma n^{-1/3})^{4/3}\\ &\geq n^{4/3}\left(1-\frac{\gamma}{c}\right)^{4/3}\\ &= \left(\frac{n}{c}\right)^{4/3}\\ &\geq nc^{-1/3},\end{aligned}$$

so that the left side of (1.13) exceeds

$$12(c^{-1/3}-2c^{-10/3}).$$

For $2 \leq c \leq 1728$ this expression is greater than 1 as required to complete the proof. ∎

Proposition 1.8. The lower bound of Proposition 1.7 is greater than or equal to the lower bound of Proposition 1.6 in all cases where both propositions are applicable.

Proof. It suffices to show that

$$\begin{aligned}&\left[1-\frac{k}{(n-k(c-1))^{2/3}}\right]\exp\left(-\frac{k^2(2c-1)}{2(n-k(c-1))}\right)\\ &\quad\geq \left[1-k\left(\frac{2c-1}{n}\right)^{2/3}\right]\exp\left(-\frac{k^2(2c-1)}{n}\right),\end{aligned}$$

or equivalently that

$$(1.14)\qquad \left[1 - \frac{k}{(n-k(c-1))^{2/3}}\right] \exp\left(-\frac{k^3(c-1)(2c-1)}{2n(n-k(c-1))}\right) \geq 1 - k\left(\frac{2c-1}{n}\right)^{2/3},$$

for all c, n, and k which satisfy the hypotheses of both propositions and are such that $k < (\frac{n}{2c-1})^{2/3}$. Letting γ denote $c-1$, the left side of (1.14) exceeds

$$\left[1 - \frac{k}{(n-k\gamma)^{2/3}}\right]\left[1 - \frac{k^3\gamma(2\gamma+1)}{2n(n-k\gamma)}\right] > 1 - \frac{k}{(n-k\gamma)^{2/3}} - \frac{k^3\gamma(2\gamma+1)}{2n(n-k\gamma)}.$$

Hence it suffices to show that

$$k\left(\frac{2\gamma+1}{n}\right)^{2/3} \geq \frac{k}{(n-k\gamma)^{2/3}} + \frac{k^3\gamma(2\gamma+1)}{2n(n-k\gamma)}.$$

The above inequality will be true if

$$(1.15)\qquad \frac{3}{4}\left(\frac{2\gamma+1}{n}\right)^{2/3} \geq \frac{1}{(n-k\gamma)^{2/3}}$$

and

$$(1.16)\qquad \frac{1}{4}\left(\frac{2\gamma+1}{n}\right)^{2/3} \geq \frac{k^2\gamma(2\gamma+1)}{2n(n-k\gamma)}.$$

Now (1.15) is true if and only if

$$\frac{k\gamma(2\gamma+1)^{2/3}}{n} \leq (2\gamma+1)^{2/3}\left[1 - \frac{8}{3^{3/2}(2\gamma+1)}\right]$$

which holds since

$$\frac{k\gamma(2\gamma+1)^{2/3}}{n} < \frac{\gamma}{n^{1/3}} < 1$$

and

$$(2\gamma+1)^{2/3}\left[1 - \frac{8}{3^{3/2}}(2\gamma+1)\right] \geq 3^{2/3}\left[1 - \frac{8}{3^{5/2}}\right] > 1.$$

Also, (1.16) holds whenever

$$\frac{1}{2k}\left(\frac{n}{2\gamma+1}\right)^{1/3}\left(\frac{n}{k\gamma} - 1\right) \geq 1.$$

To complete the proof note that the left side above exceeds

$$\frac{1}{2}\left(\frac{n^{1/3}}{\gamma}(2\gamma+1)^{2/3} - 1\right) > \frac{1}{2}\left(\frac{\gamma+1}{\gamma}(2\gamma+1)^{2/3} - 1\right)$$

which is greater than 1 whatever $\gamma \in \{1, 2, \cdots\}$ may be. ∎

The following proposition is just a special case of Proposition 1.3.

Proposition 1.9. For fixed $t > 0$,

$$P\left\{\tau_{c,n}/\sqrt{\frac{2n}{2c-1}} > t\right\} \sim e^{-t^2} \quad \text{as} \quad n \to \infty.$$

Since the upper bound furnished by Proposition 1.5 is valid over a larger range than is the upper bound given by Proposition 1.2, the result given in Proposition 1.4 may now be extended over a larger range as well. The proof of the following proposition is omitted since it is entirely similar to the proof of Proposition 1.4.

Proposition 1.10. For $0 < \alpha < \frac{2}{3}$,

$$P\left\{\tau_{c,n} > \left[\!\left[\left(\frac{n}{2c-1}\right)^{\alpha}\right]\!\right]\right\} \sim \exp\left[-\frac{1}{2}\left(\frac{n}{2c-1}\right)^{2\alpha-1}\right] \quad \text{as} \quad n \to \infty.$$

In order to investigate the accuracy of the other formulas presented thus far in this section, consider as an example the simple birthday problem where c equals 1 and n equals 365. (1.11) becomes

$$P\{\tau_{1,365} > k\} = \prod_{j=1}^{k-1}\left(1 - \frac{j}{365}\right) \quad (k = 2, \cdots, 365). \tag{1.17}$$

Proposition 1.5 gives

$$P\{\tau_{1,365} > k\} < e^{-\frac{(k-1)k}{730}} \quad (k = 2, \cdots, 365) \tag{1.18}$$

and Proposition 1.6 provides

$$P\{\tau_{1,365} > k\} \begin{cases} > e^{-\frac{k^2}{730}}\left(1 - \frac{k}{51}\right) & (k = 2, \cdots, 50) \\ \geq 0 & (k \geq 51)\,. \end{cases} \tag{1.19}$$

Proposition 1.9 suggests the often used approximation,

$$P\{\tau_{1,365} > k\} \approx e^{-\frac{k^2}{730}}. \tag{1.20}$$

Below the right hand sides of (1.17), (1.18), (1.19), and (1.20) are compared for several values of k.

k	lower bound (1.19)	exact prob. (1.17)	upper bound (1.18)	approx. (1.20)
5	0.872	0.973	0.973	0.966
10	0.701	0.883	0.884	0.872
20	0.351	0.589	0.594	0.578
30	0.120	0.294	0.304	0.291
40	0.024	0.109	0.118	0.112
50	6.4×10^{-4}	0.030	0.035	0.033
60	0	0.006	0.008	0.007

It can be seen above that the upper bound is reasonably tight, but that the lower bound is not. It may also be observed that the approximation given by (1.20) is neither an upper bound nor a lower bound. It underestimates the true probability for small values of k, and overestimates for larger k.

Putting c equal to 7 and proceeding like before yields the following results.

$$P\{\tau_{7,365} > k\} = \frac{1}{(365)^{k-1}} \prod_{j=1}^{k-1} (365 - 6k - j) \quad (k = 2, \cdots, 52). \tag{1.21}$$

$$P\{\tau_{7,365} > k\} < e^{-\frac{13k(k-1)}{730}} \quad (k = 2, \cdots, 52), \tag{1.22}$$

$$P\{\tau_{7,365} > k\} > \begin{cases} e^{-\frac{13k(k-1)}{730}} \left(1 - \frac{k}{9.23}\right) & (k = 2, \cdots, 9) \\ 0 & (k = 10, \cdots, 52) \end{cases} \tag{1.23}$$

and

$$P\{\tau_{7,365} > k\} \approx e^{-\frac{13k^2}{730}}. \tag{1.24}$$

In addition, Proposition 1.7 provides the alternate lower bound

$$P\{\tau_{7,365} > k\} > \begin{cases} e^{-\frac{13k^2}{(730-12k)}} \left(1 - \frac{k}{(365-6k)^{2/3}}\right) & (k = 2, \cdots, 30) \\ 0 & (k = 31, \cdots, 52). \end{cases} \tag{1.25}$$

A comparison of these results is presented below.

k	simple lower bound (1.23)	improved lower bound (1.25)	exact probability (1.21)	upper bound (1.22)	approx. (1.24)
3	0.606	0.794	0.896	0.899	0.852
5	0.320	0.552	0.689	0.700	0.641
10	0	0.093	0.171	0.201	0.168
20	0	1.2×10^{-5}	2.3×10^{-4}	1.2×10^{-2}	8.1×10^{-4}
30	0	1.4×10^{-15}	2.3×10^{-10}	1.9×10^{-7}	1.1×10^{-7}

Observe that the upper bound again outperforms the lower bounds, and also that the approximation formula behaves similar to the one in the previous example. It is interesting to note that as poor as the simple lower bound of Proposition 1.6 (or equivalently of the corollary following Lemma 1.1) appears to be, it is tight enough to help establish the asymptotic formula for $E[\tau]$ stated in Theorem 1.2, which follows. The looseness of the lower bound arises from the fact that the linear factor $\left[1 - k\left(\frac{2c-1}{n}\right)^{\alpha}\right]$ severely overcompensates for the tail probabilities decaying faster than the exponential factor $\exp\left(-k^2 \frac{(2c-1)}{2n}\right)$.

Lower bounds for the exact probabilities in the two examples above may also be produced from the corollary following Lemma 1.2. The values obtained (which are not reported here) are at least as good as those given by (1.19) and (1.23) for all values of k; however, while the values gotten from the corollary are better than those given by (1.25) for some values of k, they are worse for other values of k.

Now a result for the mean will be derived using previous results for the distribution of τ.

Theorem 1.2. For a packing sequence consisting of equivalent points, if

$$P\{\tau > k\} \leq e^{-\frac{(k-1)^2}{2}p}$$

for $k \geq 2$, then

$$E[\tau] \sim \sqrt{\frac{\pi}{2p}} \quad \text{as} \quad p \downarrow 0.$$

Proof. By Theorem 1.1

$$\lim_{p \to 0^+} \frac{E[\tau]}{\sqrt{\frac{\tau}{2p}}} \geq 1. \tag{1.26}$$

Recalling from the proof of Theorem 1.1 that

$$E[\tau] = 2 + \sum_{k=2}^{\infty} P\{\tau > k\},$$

it follows that if $P\{\tau > k\} \leq \exp\left[-\frac{(k-1)^2}{2}p\right]$ then

$$\begin{aligned}
\lim_{p \to 0^+} \frac{E[\tau]}{\sqrt{\frac{\pi}{2p}}} &\leq \lim_{p \to 0^+} \frac{\left(2 + \sum_{k=1}^{\infty} \epsilon^{-k^2/2p}\right)}{\sqrt{\frac{\tau}{2p}}} \\
&= \lim_{p \to 0^+} \sqrt{\frac{2p}{\pi}} \sum_{k=1}^{\infty} \epsilon^{-\frac{k^2}{2}p} \\
&\leq \lim_{p \to 0^+} \sqrt{\frac{2p}{\pi}} \int_0^{\infty} \epsilon^{-\frac{k^2}{2}p} dx \\
&= \lim_{p \to 0^+} 1 \\
&= 1.
\end{aligned}$$

This result and (1.26) together imply

$$\lim_{p \to 0^+} \frac{E[\tau]}{\sqrt{\frac{\tau}{2p}}} = 1$$

as required to complete the proof. ∎

Corollary. $E[\tau_{c,n}] \sim \sqrt{\frac{n\pi}{2(2c-1)}}$ as $n \to \infty$.

Another method may also be used to obtain an asymptotic result concerning $E[\tau_{c,n}]$. Asymptotic results for the mean are also found in [6] and [31].

Let the sequence of random variables $\{\tau_n'\}_{n=2}^{\infty}$ be defined by

$$\tau_n' = \frac{\tau_{c,n}}{\sqrt{n}} \quad (n = 2, 3, \cdots).$$

It follows from Proposition 1.5, Proposition 1.6 and calculus that for any $\alpha > 0$

$$\lim_n P\{\tau_n' > \alpha\} = e^{-\gamma\alpha^2}$$

where $\gamma = \frac{2c-1}{2}$. Hence τ_n' converges in law to a random variable having Maxwell's distribution. Since

$$\begin{aligned}
&\lim_{\alpha\to\infty} \sup_n \int_{[\tau_n' \geq \alpha]} \tau_n' dP \\
&\leq \lim_{\alpha\to\infty} \sup_n \left(\alpha \exp\left(-\alpha^2 + \frac{3\alpha}{\sqrt{n}} - \frac{2}{n}\right) + \int_{(\alpha - \frac{3}{\sqrt{n}})}^{\infty} e^{-\gamma x^2} dx\right) \\
&= \lim_{\alpha\to\infty} \left(\alpha \exp\left(-\alpha^2 + \frac{3\alpha}{\sqrt{2}}\right) + \int_{(\alpha - \frac{3}{\sqrt{2}})}^{\infty} e^{-\gamma x^2} dx\right) \\
&= 0.
\end{aligned}$$

the random variables τ_n' are also uniformly integrable. Thus it now follows (see [3]) that $E[\tau_n']$ converges to the mean of Maxwell's distribution, i.e.

$$E[n^{-1/2}\tau_{c,n}] \to \int_0^\infty x e^{-\gamma x^2} dx = \sqrt{\frac{\pi}{2(2c-1)}}.$$

1.4. Random Arcs On a Circle

In the setting of the previous section, if c is taken to be very large and n considerably larger, then the packing sequence of placing arcs of length c uniformly on a circle of length n so that their endpoints have integer coordinates is very nearly the same as placing arcs of length $\frac{c}{n}$ on a circle of unit circumference with their midpoints chosen according to a uniform $(0, 1]$ distribution. So in a sense, collision problems for random arcs on a circle are just continuous versions of birthday problems, with one of the more basic problems being to determine the probability that a set of k random arcs will be pairwise disjoint.

In terms of the notation of Section 1, the packing sequence for arcs of length a on a circle corresponds to putting $S = (0, 1]$,

$$\mu(x, y) = \begin{cases} |x - y| & \text{if } |x - y| \leq \frac{1}{2} \\ 1 - |x - y| & \text{otherwise,} \end{cases}$$

and $\delta = a$. The packing sequence has equivalent points.

The distribution of the time to the initial collision, τ_a, may be obtained from the work of Stevens [57]. It is found that for $0 < a < 1$ and k an integer,

$$P\{\tau_a > k\} = \begin{cases} 1 & \text{if } k \le 1 \\ (1 - ka)^{k-1} & \text{if } 1 < k \le [a^{-1}] \\ 0 & \text{if } k > [a^{-1}]. \end{cases}$$

The essential step in Stevens' proof is the establishment of a one to one correspondence between configurations of k arcs having no overlaps with configurations of arcs for which none of the endpoints $C_2, C_3, \cdots, C_k$ lie in a specified region of length ka. Since the probability that $k - 1$ random points fall outside of a region of length ka is just

$$(1 - ka)^{k-1},$$

the above formula follows.

In contrast to the unwieldy product formula of the previous section, the expression $(1 - ka)^{k-1}$ can be easily computed with a hand held calculator even if k is very large. Nevertheless, for the purpose of establishing asymptotic results it is convenient to derive upper and lower bounds for $P\{\tau_a > k\}$.

It should also be mentioned that the asymptotics of Stevens' formula have been thoroughly studied because of its connections with time series. One such interesting connection is due to the following fact:

$$P\{\tau_a > k\} = \left(P\left\{\frac{X_1}{S_k} > a\right\}\right)^{k-1},$$

where $X_1, \cdots, X_k$ are i.i.d. exponential random variables and $S_k = \sum_{i=1}^{k} X_i$.

Proposition 1.11. For $a \in (0, \frac{1}{2})$ and k an integer satisfying $2 \le k < a^{-1}$,

$$P\{\tau_a > k\} < e^{-ak(k-1)}. \tag{1.27}$$

$$P\{\tau_a > k\} > e^{-\frac{ak(k-1)}{(1-ka)}}. \tag{1.28}$$

and, with the additional stipulation that $a \le \frac{1}{8}$,

$$P\{\tau_a > k\} > \begin{cases} \epsilon^{-ak(k-1)}(1 - a^{2/3}k) & \text{if } k < a^{-2/3} \\ 0 & \text{otherwise.} \end{cases} \tag{1.29}$$

Further, the lower bound (1.28) is greater than the lower bound (1.29).

Proof. The double inequality

$$(1-x)^{\frac{1}{x}} < \epsilon^{-1} < (1-x)^{\left(\frac{1-x}{x}\right)} \quad (0 < x < 1)$$

may be used to establish (1.27) and (1.28) as follows:

$$\begin{aligned} P\{\tau_a > k\} &= (1-ka)^{k-1} \\ &= [(1-ka)^{\frac{1}{ka}}]^{ka(k-1)} \\ &< \epsilon^{-ak(k-1)} \end{aligned}$$

and

$$\begin{aligned} P\{\tau_a > k\} &= \left[(1-ka)^{\left(\frac{1-ka}{ka}\right)}\right]^{\frac{ka(k-1)}{(1-ka)}} \\ &> \epsilon^{-\frac{ak(k-1)}{(1-ka)}}. \end{aligned}$$

To prove (1.29) it is enough to show that for all appropriate k and a,

$$\log\left\{\epsilon^{-\frac{ak(k-1)}{(1-ka)}}[\epsilon^{-ak(k-1)}(1-a^{2/3}k)]^{-1}\right\} \ge 0.$$

Using (1.3) it follows that the left side above exceeds

$$\begin{aligned} &-\frac{(k-1)(ka)^2}{(1-ka)} + a^{2/3}k + \frac{1}{2}a^{4/3}k^2 + \frac{1}{3}a^2k^3 + \frac{1}{4}a^{8/3}k^4 \\ &> -\frac{k^3a^2}{1-a^{1/3}} + a^{2/3}k + \frac{1}{2}a^{4/3}k^2 + \frac{1}{3}a^2k^3 + \frac{1}{4}a^{8/3}k^4 \\ &> a^{2/3}k + \frac{1}{2}a^{4/3}k^2 - \frac{5}{3}a^2k^3 + \frac{1}{4}a^{8/3}k^4. \end{aligned}$$

It now suffices to have

$$\frac{1}{2}a^{2/3}k - \frac{5}{3}a^{4/3}k^2 + \frac{1}{4}a^2k^3 > -1$$

for all k such that $2 \le k < a^{-2/3}$. It is not hard to show that all points on the curve

$$y = \frac{1}{4}x^3 - \frac{5}{3}x^2 + \frac{1}{2}x$$

for which $x \in (0,1)$ lie above the line $y = -1$, which is sufficient to complete the proof of (1.29). Note that the claim following(1.29) has been proven as well. ∎

The proof of the following proposition is omitted since the results are easily established using (1.27) and (1.29). Also (i) is just a special case of Proposition 1.3.

Proposition 1.12. (i) For any integer $t > 0$,

$$P\{a^{\frac{1}{2}}\tau_a > t\} \sim e^{-t^2} \quad \text{as} \quad a \downarrow 0.$$

(ii) For any $\alpha \in (0, 2/3)$,

$$P\{\tau_a > [\![a^{-\alpha}]\!]\} \sim e^{-a^{1-2\alpha}} \quad \text{as} \quad a \downarrow 0.$$

It is interesting to note that whether (1.29) or the more accurate lower bound (1.28) is used to prove (ii) above, the requirement that α be less than $\frac{2}{3}$ is necessary. An examination of the following expansion is helpful in understanding why this is so. For $k < a^{-1}$

$$\begin{aligned}(1-ka)^{k-1} &= \exp\{(k-1)\log(1-ka)\}\\ &= \exp\left\{(k-1)\left[-ka - \frac{(ka)^2}{2} - \frac{(ka)^3}{3} - \cdots\right]\right\}\\ &= e^{-(k-1)ka}\exp\left\{-(k-1)\frac{(ka)^2}{2} - (k-1)\frac{(ka)^3}{3} - \cdots\right\}.\end{aligned}$$

It then follows that

$$P\{\tau_a > [\![a^{-\alpha}]\!]\} \sim e^{-a^{1-2\alpha}}\exp\left\{-\frac{a^{2-3\alpha}}{2}\right\} \quad \text{as} \quad a \downarrow 0$$

for all $\alpha < \frac{3}{4}$. From this last expression it can be seen that the statement in part (ii) of Proposition 1.12 is true for any $0 < \alpha < \frac{2}{3}$. Furthermore, it is apparent from the expansion above that the statement is not true for any $\alpha \geq \frac{2}{3}$.

With $k_\alpha(a)$ defined to be the nearest integer to $a^{-\alpha}$, i.e. $k_\alpha(a) = [\![a^{-\alpha} + \frac{1}{2}]\!]$, Table A compares $P\{\tau_a > k_\alpha(a)\}$ and $\exp(-ak_\alpha^2(a))$ for various choices of a and α. The ratio $\exp(-ak_\alpha^2(a))/P\{\tau_a > k_\alpha(a)\}$ is denoted by $r_\alpha(a)$.

Table A

	a	$k_\alpha(a)$	$(1 - ak_\alpha(a))^{k_\alpha(a)-1}$	$e^{-ak_\alpha^2(a)}$	$r_\alpha(a)$
	10^{-2}	10	0.387	0.368	0.950
	10^{-3}	32	0.365	0.359	0.984
$\alpha = \frac{1}{2}$	10^{-4}	100	0.370	0.368	0.995
	10^{-5}	316	0.369	0.368	0.998
	10^{-6}	1000	0.368	0.368	0.999
	10^{-2}	16	7.31×10^{-2}	7.73×10^{-2}	1.06
	10^{-3}	63	1.77×10^{-2}	1.89×10^{-2}	1.07
$\alpha = \frac{3}{5}$	10^{-4}	251	1.74×10^{-3}	1.84×10^{-3}	1.06
	10^{-5}	1000	4.36×10^{-5}	4.54×10^{-5}	1.04
	10^{-6}	3981	1.27×10^{-7}	1.31×10^{-7}	1.03
	10^{-2}	18	3.43×10^{-2}	3.92×10^{-2}	1.14
	10^{-3}	75	3.12×10^{-3}	3.61×10^{-3}	1.16
$\alpha = \frac{5}{8}$	10^{-4}	316	4.05×10^{-5}	4.61×10^{-5}	1.14
	10^{-5}	1334	1.68×10^{-8}	1.87×10^{-8}	1.11
	10^{-7}	23714	3.54×10^{-25}	3.78×10^{-25}	1.07
	10^{-2}	22	5.42×10^{-3}	7.91×10^{-3}	1.46
$\alpha = \frac{2}{3}$	10^{-3}	100	2.95×10^{-5}	4.54×10^{-5}	1.54
	10^{-4}	464	2.80×10^{-10}	4.46×10^{-10}	1.60
	10^{-7}	46416	1.65×10^{-94}	2.71×10^{-94}	1.64
	10^{-2}	25	1.00×10^{-3}	1.93×10^{-3}	1.92
$\alpha = \frac{7}{10}$	10^{-3}	126	4.89×10^{-8}	1.27×10^{-7}	2.61
	10^{-4}	631	1.47×10^{-18}	5.11×10^{-18}	3.48
$\alpha = \frac{3}{4}$	10^{-2}	32	6.42×10^{-6}	3.57×10^{-5}	5.56
	10^{-3}	178	8.56×10^{-16}	1.74×10^{-14}	20.3

The results shown in the table suggest that the approximation formula

$$P\{\tau_a > k\} \approx e^{-ak^2}$$

works rather well for $k \leq a^{-0.6}$, and also performs reasonably for the case of $\alpha = 0.625$. The approximation seems somewhat disastrous for $\alpha \geq \frac{2}{3}$; however, the case of $\alpha = \frac{2}{3}$ is not nearly as severe as the cases for which $\alpha > \frac{2}{3}$. Similar results for cases with $\alpha = 0.65$ and $\alpha = 0.66$ have the ratio $r_o(a)$ tending slowly to 1 as expected. Thus the requirement that α be less than $\frac{2}{3}$ in part (ii) of Proposition 1.12 doesn't seem to be needed only because a sufficiently tight lower bound is not known for $P\{\tau_a > k\}$, rather its necessity seems to reflect the apparent fact that the deviation from exponentiality in the tail of the distribution of τ_a tends to become more pronounced for values around $a^{-2/3}$ or greater.

The following proposition follows immediately from (1.27) and Theorem 1.2 since for random arcs of length a on a circle of circumference 1, $p = 2a$.

Proposition 1.13. $E[\tau_a] \sim \frac{1}{2}\sqrt{\frac{\pi}{a}}$ as $a \downarrow 0$.

The results of this section may also be applied to the problem of spacings on the unit circle or, if end effects are ignored, on the unit interval. Suppose that $C_1, C_2, \cdots, C_k$ are selected at random from $(0, 1]$. Form the order statistics

$$C_{(1)} \leq C_{(2)} \leq \cdots \leq C_{(k)},$$

and define the spacings $S_1, S_2, \cdots, S_k$ by

$$S_j = \begin{cases} C_{(1)} - C_{(n)} + 1 & \text{for } j = 1 \\ C_{(j)} - C_{(j-1)} & \text{for } j = 2, \cdots, k. \end{cases}$$

Let

$$M_k = \min\{S_1, S_2, \cdots, S_k\}.$$

Then

$$P\{M_k \geq a\} = P\{\tau_a > k\}.$$

As an example, it may be concluded from the results in Table A that the approximation

$$P\{M_k \geq a\} \approx e^{-ak^2}$$

should be fairly good whenever a is less than $\min\{k^{-5/3}, 10^{-2}\}$.

Holst [22] obtains several nice results concerning the random variable M_k. He also examines the length of the jth smallest spacing, a quantity which is related to the probability of j collisions occurring. In [23], Holst studied the asymptotic behavior of the distribution of the largest spacing, which is directly related to the probability that a circle is completely covered by random arcs of equal length. A review of other results concerning spacings is contained in [46].

1.5. Arcs of Unequal Length

This section deals with sequences of random arcs having fixed, but not necessarily equal, lengths. The arcs have clockwise endpoints $C_1, C_2, \cdots$ which are independently selected from a circle of unit circumference according to a uniform distribution. The first arc has fixed length a_1, the second arc has length a_2, and so on.

In this setting, two arcs will collide whenever any portion of them overlap. Note that the dual interpretation of saying that a collision occurs whenever two selections from the space are separated by a distance less than some fixed $\delta > 0$ is no longer applicable.

Letting τ denote the time to the first overlap, it is easy to see that for $a_1 + a_2 \leq 1$

$$P\{\tau > 2\} = 1 - a_2 - a_2.$$

A simple inclusion-exclusion argument yields that if $a_1 + a_2 + a_3 \leq 1$ then

$$P\{\tau > 3\} = (1 - a_1 - a_2 - a_3)^2.$$

Likewise, for $a_1 + a_2 + a_3 + a_4 \leq 1$

$$P\{\tau > 4\} = (1 - a_1 - a_2 - a_3 - a_4)^3.$$

The calculation of this last result, or any subsequent result, by brute force is extremely tedious; however, a slight modification of a clever argument due to Stevens [57] establishes that

$$P\{\tau > k\} = [(1 - s_k)_+]^{k-1} \tag{1.30}$$

for $k \geq 2$, where $s_k = a_1 + \cdots + a_k$. (1.30) may also be obtained by appealing to a result due to Marsaglia [39] and de Finetti [8]. Their result states that if $(X_1, X_2, \cdots, X_n)$ is a random point on the simplex $\{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0, x_1 + \cdots + x_n = 1\}$, then

$$P\{X_1 \geq a_1, \cdots, X_n \geq a_n\} = [(1 - a_1 - \cdots - a_n)_+]^{n-1}.$$

(1.30) follows immediately since it can be shown that the joint distribution of the set of spacings which occur between n points independently and uniformly selected from a circle of unit circumference is uniform on the simplex given above (see p. 76 of Feller [15]).

An interesting consequence of (1.30) is that $P\{\tau = k\}$ depends only on a_k and s_{k-1} and not on the individual values of $a_1, \cdots, a_{k-1}$. This may be seen through the equation

$$\begin{aligned} P\{\tau = k\} &= P\{\tau > k-1\} - P\{\tau > k\} \\ &= (1 - s_{k-1})^{k-2} - (1 - s_{k-1} - a_k)^{k-1}. \end{aligned}$$

τ can be made to have numerous interesting distributions by choosing $a_1, a_2, \cdots$ in certain ways. In the examples given below, several cases are examined in which the sequence of arc lengths is either monotonically increasing or decreasing. In the first case presented, the prescribed sequence of lengths has the time of the first collision being equally likely over a range of values.

Uniform Collision Time

If $a_1, a_2, \cdots, a_m$ are such that

$$\begin{aligned} a_1 + a_2 &= (m-1)^{-1}, \\ a_k &= \left[1 - \left(\frac{k-2}{m-1}\right)\right]^{\frac{1}{(k-2)}} - \left[1 - \left(\frac{k-1}{m-1}\right)\right]^{\frac{1}{(k-1)}} \end{aligned}$$

for $k = 3, 4, \cdots, m-1$, and

$$a_m \geq \left(\frac{1}{m-1}\right)^{\frac{1}{(m-2)}},$$

then

$$P\{\tau = 2\} = P\{\tau = 3\} = \cdots = P\{\tau = m\} = \frac{1}{m-1}$$

and

$$E[\tau] = \frac{m}{2} + 1.$$

Geometrically Decreasing Arcs

Now consider the sequence of arc lengths given by

$$a_k = \gamma^{k-1} a$$

where $\gamma \in (0, 1)$ and $0 < a \leq \frac{1}{2}$. For $k \geq 2$,

$$s_k = \left(\frac{1 - \gamma^k}{1 - \gamma} \right) a$$

so that

$$P\{\tau > k\} = \left[1 - \left(\frac{1 - \gamma^k}{1 - \gamma} \right) a \right]^{k-1} \qquad (k = 2, 3, \cdots).$$

Some asymptotic results for this setting are given in the following theorem.

Theorem 1.3. If $a_k = \gamma^{k-1} a$ where $\gamma \in (0, 1)$ and $0 < a \leq \frac{1}{2}$, then

(i) $P\{\tau \leq k\} \sim (k-1) \left(\frac{1-\gamma^k}{1-\gamma} \right) a$ as $a \downarrow 0$

and

(ii) $E[\tau] \sim \frac{1-\gamma}{a}$ as $a \downarrow 0$.

Proof. The following two bounds can clearly establish (i).

$$\begin{aligned} P\{\tau \leq k\} &= 1 - \left[1 - \left(\frac{1 - \gamma^k}{1 - \gamma} \right) a \right]^{k-1} \\ &< (k-1) \left(\frac{1 - \gamma^k}{1 - \gamma} \right) a, \end{aligned}$$

and for $a \leq 4 \left[(k-1) \left(\frac{1-\gamma^k}{1-\gamma} \right) \right]^{-1}$

$$\begin{aligned} P\{\tau \leq k\} &= 1 - \exp \left\{ (k-1) \log \left[1 - \left(\frac{1 - \gamma^k}{1 - \gamma} \right) a \right] \right\} \\ &> 1 - \exp \left\{ -(k-1) \left(\frac{1 - \gamma^k}{1 - \gamma} \right) a \right\} \\ &> (k-1) \left(\frac{1 - \gamma^k}{1 - \gamma} \right) a - \frac{1}{2} \left[(k-1) \left(\frac{1 - \gamma^k}{1 - \gamma} \right) \right]^2 a^2. \end{aligned}$$

Now (ii) will be established.

$$
\begin{aligned}
E[\tau] &= \sum_{k=0}^{\infty} P\{\tau > k\} \\
&= 2 + \sum_{k=2}^{\infty} \left[1 - \left(\frac{1-\gamma^k}{1-\gamma}\right) a\right]^{k-1} \\
&> 2 + \sum_{k=2}^{\infty} \left(1 - \frac{a}{1-\gamma}\right)^{k-1} \\
&= 1 + \frac{1-\gamma}{a},
\end{aligned}
$$

and so

$$
\liminf_{a\to 0^+} \frac{E[\tau]}{\left(\frac{1-\gamma}{a}\right)} \geq 1. \tag{1.31}
$$

Also, for any $m \geq 3$ and a sufficiently small,

$$
\begin{aligned}
E[\tau] &= 2 + \sum_{k=2}^{m-1} \left[1 - \left(\frac{1-\gamma^k}{1-\gamma}\right) a\right]^{k-1} + \sum_{k=m}^{\infty} \left[1 - \left(\frac{1-\gamma^k}{1-\gamma}\right) a\right]^{k-1} \\
&< 2 + (m-2) + \sum_{k=m}^{\infty} \left[1 - \left(\frac{1-\gamma^m}{1-\gamma}\right) a\right]^{k-1} \\
&= m + \frac{\left[1 - \left(\frac{1-\gamma^m}{1-\gamma}\right) a\right]^{m-1}}{\left(\frac{1-\gamma^m}{1-\gamma}\right) a} \\
&< m + \left(\frac{1-\gamma}{1-\gamma^m}\right) \frac{1}{a}
\end{aligned}
$$

so that

$$
\limsup_{a\to 0^+} \frac{E[\tau]}{\left(\frac{1-\gamma}{a}\right)} \leq \frac{1}{1-\gamma^m}.
$$

Since the above inequality holds for all $m \geq 3$, it must be true that

$$
\limsup_{a\to 0^+} \frac{E[\tau]}{\left(\frac{1-\gamma}{a}\right)} \leq 1.
$$

This result and (1.31) together imply that

$$
E[\tau] \sim \frac{1-\gamma}{a} \quad \text{as} \quad a \to 0^+. \qquad \blacksquare
$$

Displayed below are a few exact values of $E[\tau]$ along with their corresponding asymptotic approximations for the case of $\gamma = \frac{1}{2}$. Note that the lower bound $1 + \frac{1-\gamma}{a}$ provides

an estimate of $E[\tau]$ which is even more precise than the asymptotic form $\frac{1-\gamma}{a}$ given by the theorem above.

a	$E[\tau]$	$\frac{1-\gamma}{a}$
0.1	6.1401	5
0.01	51.019	50
0.002	251.00	250

Uniformly Increasing Arcs

Now instead of a decreasing sequence of arc lengths, consider the case where the lengths increase according to

$$a_k = ka.$$

Then

$$P\{\tau > k\} = \begin{cases} \left[1 - \frac{k(k+1)}{2}a\right]^{k-1} & (2 \le k < M(a)) \\ 0 & (k \ge M(a)) \end{cases}$$

where

$$M(a) = \left\lceil \frac{\sqrt{1+\frac{8}{a}} - 1}{2} \right\rceil .$$

Asymptotic results concerning τ are provided by the following theorem.

Theorem 1.4. If $a_k = ka$, then

(i) $P\left\{\left(\frac{a}{2}\right)^{1/3}\tau > t\right\} \sim e^{-t^3}$ as $a \downarrow 0$

and

(ii) $E[\tau] \sim \left(\frac{2}{a}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)$ as $a \downarrow 0$.

Proof. For $k < M(a)$

$$\begin{aligned} \log P\{\tau > k\} &= (k-1)\log\left[1 - \frac{k(k+1)}{2}a\right] \\ &< -\frac{(k-1)k(k+1)}{2}a \\ &< -\frac{(k-1)^3}{2}a. \end{aligned}$$

so that

$$P\{\tau > k\} < \exp\left[-\frac{(k-1)^3}{2}a\right] \qquad (2 \leq k < M(a)).$$

Then for a sufficiently small

$$\begin{aligned} P\left\{\left(\frac{a}{2}\right)^{1/3}\tau > t\right\} &= P\left\{\tau > \left[\left[\left(\frac{2}{a}\right)^{1/3} t\right]\right]\right\} \\ &< \exp\left\{-\frac{a}{2}\left(\left[\left[\left(\frac{2}{a}\right)^{1/3} t\right]\right] - 1\right)^3\right\} \\ &< \exp\left\{-\frac{a}{2}\left(\left(\frac{2}{a}\right)^{1/3} t - 2\right)^3\right\} \\ &= \exp\left\{-t^3 + 6\left(\frac{a}{2}\right)^{1/3} t^2 - 12\left(\frac{a}{2}\right)^{2/3} t + 4a\right\}, \end{aligned}$$

so that

$$\limsup_{a\to 0^+} \frac{P\{(\frac{a}{2})^{1/3}\tau > t\}}{e^{-t^3}} \leq 1.$$

If $a \leq \min\left\{\frac{6}{5k(k+1)}, \sqrt{\frac{12}{k^2(k+1)^2(k-1)}}\right\}$, then

$$\begin{aligned} P\{\tau > k\} &= \exp\left\{(k-1)\log\left[1 - \frac{k(k+1)}{2}a\right]\right\} \\ &> \exp\left\{(k-1)\left[-\frac{k(k+1)}{2}a - \left(\frac{k(k+1)}{2}a\right)^2\right]\right\} \\ &> \epsilon^{-\frac{k^3}{2}a}\left[1 - (k-1)\left(\frac{k(k+1)}{2}a\right)^2\right] \\ &> \epsilon^{-\frac{k^3}{2}a}\left[1 - \frac{k^5}{2}a^2\right]. \end{aligned}$$

It follows that if a is sufficiently small then

$$\begin{aligned} P\left\{\left(\frac{a}{2}\right)^{1/3}\tau > t\right\} &= P\left\{\tau > \left[\left[\left(\frac{2}{a}\right)^{1/3} t\right]\right]\right\} \\ &> \exp^{-t^3}\left[1 - (4a)^{1/3} t^5\right]. \end{aligned}$$

Therefore

$$\liminf_{a\to 0+} \frac{P\{(\frac{a}{2})^{1/3}\tau > t\}}{e^{-t^3}} \geq 1,$$

which when combined with a previous result establishes (i).

The bounds found above yield that

$$
\begin{aligned}
E[\tau] &= 2 + \sum_{k=2}^{M(a)-1} P\{\tau > k\} \\
&< 2 + \sum_{k=2}^{M(a)-1} \epsilon^{-\frac{(k-1)^3}{2} a} \\
&< 2 + \int_0^\infty e^{-\frac{a}{2}x^3}\, dx \\
&= 2 + \left(\frac{2}{a}\right)^{1/3} \Gamma\left(\frac{4}{3}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
E[\tau] &= 2 + \sum_{k=2}^{M(a)-1} P\{\tau > k\} \\
&> \int_0^{M(a)} \epsilon^{-\frac{x^3}{2} a} \left(1 - \frac{x^5}{2} a^2\right) dx.
\end{aligned}
$$

Now for a sufficiently small, $\frac{x^5}{2} a^2 > 1$ for all $x > M(a)$. Therefore, when a is small enough,

$$
\begin{aligned}
E[\tau] &> \int_0^\infty \epsilon^{-\frac{x^3}{2} a} \left(1 - \frac{x^5}{2} a^2\right) dx \\
&= \left(\frac{2}{a}\right)^{1/3} \Gamma\left(\frac{4}{3}\right) - \frac{2}{3}.
\end{aligned}
$$

These upper and lower bounds for $E[\tau]$ can be used to easily establish (ii). ∎

The accuracy of the approximation

$$
E[\tau] \approx \left(\frac{2}{a}\right)^{1/3} \Gamma\left(\frac{4}{3}\right)
$$

is demonstrated below for various values of a.

a	$E[\tau]$	$\left(\frac{2}{a}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)$
$\frac{1}{15}$	3.197	2.775
$\frac{1}{55}$	4.672	4.279
$\frac{1}{325}$	8.102	7.736
1.98×10^{-4}	19.65	19.30
4.98×10^{-5}	30.93	30.59
7.98×10^{-6}	56.63	56.29
2.00×10^{-6}	89.66	89.33
5.00×10^{-7}	142.1	141.8
2.22×10^{-7}	186.1	185.8

Quadratic Growth

If the arc lengths are given by

$$a_k = k^2 a$$

then

$$P\{\tau > k\} = \left(\left[1 - \frac{k(k+1)(2k+1)}{6}a\right]_+\right)^{k-1}$$

for $2 \leq k$. For a not too large, the following bounds are true:

$$P\{\tau > k\} < \exp\left(-\frac{a}{3}k^4\right)$$

and

$$P\{\tau > k\} > \exp\left(-\frac{a}{3}(k+1)^4\right)\left[1 - \frac{a^2}{9}(k+1)^7\right].$$

These bounds can be used to prove the following theorem (whose proof is omitted) by a method which is entirely similar to that which is used in the proof of Theorem 1.4.

Theorem 1.5. If $a_k = k^2 a$, then

(i) $P\left\{\left(\frac{a}{3}\right)^{1/4}\tau > t\right\} \sim e^{-t^4}$ as $a \downarrow 0$

and

(ii) $E[\tau] \sim \left(\frac{3}{a}\right)^{1/4}\Gamma\left(\frac{5}{4}\right)$ as $a \downarrow 0$.

Displayed below are a few exact values of $E[\tau]$ along with their corresponding approximations.

a	$E[\tau]$	$(\frac{a}{3})^{-\frac{1}{4}}\Gamma(\frac{5}{4})$
3.48×10^{-4}	9.003	8.731
2.96×10^{-6}	29.03	28.77
3.00×10^{-9}	161.5	161.2

Exponentially Growing Arcs

As a final example, consider the sequence of arc lengths given by

$$a_k = \gamma^{k-1}a \qquad (\gamma > 1),$$

so that s_k increases geometrically. Then, for $k \geq 2$

$$P\{\tau > k\} = \left(\left[1 - \left(\frac{\gamma^k - 1}{\gamma - 1}\right)a\right]_+\right)^{k-1}.$$

Neither a tight upper bound nor a tight lower bound is readily established for $E[\tau]$; however, it is found that $E[\tau]$ can be closely approximated by a quadratic polynomial in $\log a$ whose coefficients depend on γ. For instance, if $\gamma = 2$ then

$$E[\tau] \approx f_2(a) = (8.811 \times 10^{-4})(\log a)^2 - 1.335 \log a - 3.10.$$

and if $\gamma = 3/2$, then

$$E[\tau] \approx f_{3/2}(a) = (5.065 \times 10^{-3})(\log a)^2 - 2.107 \log a - 6.36.$$

The closeness of these approximations can be seen in the results given below. In each case the fit was done using more than twenty points, and the comparisons presented below are just representative samples given to show the accuracy of the approximation

a	$E[\tau]$	$f_2(a)$
9.78×10^{-4}	7.14	6.20
9.54×10^{-7}	15.79	15.58
9.31×10^{-10}	25.08	25.05
9.09×10^{-13}	34.60	34.60
8.88×10^{-16}	44.23	44.23

a	$E[\tau]$	$f_{3/2}(a)$
2.61×10^{-6}	21.64	21.57
3.43×10^{-7}	26.15	26.13
4.52×10^{-8}	30.73	30.73
5.95×10^{-9}	35.37	35.37
7.84×10^{-10}	40.05	40.05

1.6. Arcs of Variable Length

This section considers another generalization of the packing sequence described in Section 4. Here each selection from the space $S = (0, 1]$ will be taken to be the clockwise endpoint of an arc having variable length on a circle of unit circumference. The length of each arc placed will be an independent observation from a distribution F, and a collision will occur whenever any pair of arcs are not disjoint. The related covering problem, where random arcs of variable length are placed on a circle until its circumference is completely covered has been studied by several investigators (see [24] and [51] for example); however, the collision problem discussed here does not seem to have received much attention.

Let the sequence of arcs be labeled $A_1, A_2, \cdots$, and let arc A_i have clockwise endpoint C_i and length L_i. Thus $C_1, C_2, \cdots$ are i.i.d. random variables with a uniform $(0, 1]$ distribution and $L_1, L_2, \cdots$ are i.i.d. random variables having c.d.f. F. Furthermore, the L_i's are independent of the C_i's.

Now let

$$S_k = L_1 + \cdots + L_k$$

and let F_k denote the distribution of S_k. Let m_F denote the mean of a random variable having distribution F, and as was the case in the previous section let τ give the time of the first overlapping of arcs.

The following theorem provides a lower bound for $P\{\tau > k\}$ in the setting described above.

Theorem 1.6. For a sequence of random arcs having lengths $L_1, L_2, \cdots \overset{\text{iid}}{\sim} F$,

$$P\{\tau > k\} \geq [(1 - km_F)_+]^{k-1}$$

for any choice of F.

Proof. It follows from (1.30) that for $k \geq 2$

$$\begin{aligned} P\{\tau > k\} &= \int P\{\tau > k \mid S_k = s\} dF_k(x) \\ &= \int [(1 - s)_+]^{k-1} dF_k(s). \end{aligned} \tag{1.32}$$

Then using Jensen's inequality, it follows from (1.32) that

$$\begin{aligned} P\{\tau > k\} &\geq [(1 - E[S_k])_+]^{k-1}, \\ &= [(1 - km_F)_+]^{k-1}. \qquad \blacksquare \end{aligned} \tag{1.33}$$

Note that a consequence of (1.33) is that $E[\tau]$ for a packing sequence of arcs having variable length prescribed by F is greater than or equal to $E[\tau]$ for a packing sequence of arcs all having length m_F.

Unfortunately, (1.32) is not necessarily easy to evaluate exactly, since for many choices of F the distribution F_k is troublesome to obtain. However, notice that if $F\left(\frac{1}{2}\right) = 1$ then

$$\begin{aligned} P\{\tau > 2\} &= E[1 - S_2] \\ &= 1 - 2E[L_1]. \end{aligned}$$

and if $F\left(\frac{1}{3}\right) = 1$ then

$$P\{\tau > 3\} = E[(1 - S_3)^2]$$
$$= [1 - 3E(L_1)]^2 + 3\,\mathrm{Var}(L_1).$$

Similarly, if L_1 is less than $\frac{1}{4}$ with probability 1, then

$$P\{\tau > 4\} = [1 - 4E(L_1)]^3 + 12\,\mathrm{Var}(L_1)[1 - 4E(L_1)] - 4E[L_1 - E(L_1)]^3.$$

Note that if the distribution F is symmetric about its mean, then the last expression simplifies to

$$P\{\tau > 4\} = [1 - 4E(L_1)]^3 + 12\,\mathrm{Var}(L_1)[1 - 4E(L_1)].$$

For larger values of k, $P\{\tau > k\}$ can also be expressed in terms of moments of L_1; however, the difficulty of doing so increases with k.

Some Examples

If F is the uniform distribution over $(a, b]$, the expressions above yield that

$$P\{\tau > 2\} = 1 - (b + a),$$
$$P\{\tau > 3\} = \left[1 - \frac{3}{2}(b + a)\right]^2 + \frac{(b - a)^2}{4},$$

and

$$P\{\tau > 4\} = [1 - 2(b + a)]^3 + (b - a)^2[1 - 2(b + a)],$$

provided that $b \le \frac{1}{4}$. For the class of uniform distributions having mean $m \le \frac{1}{8}$, the above probabilities are maximized by choosing $(a, b] = (0, 2m]$.

If F is the uniform distribution over $(0, b]$, then $P\{\tau > k\}$ may be obtained directly from (1.32) for any values of k and b. Feller [15] has shown in this case that for all s,

$$F_k(s) = \frac{1}{b^k k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} [(s - jb)_+]^k$$

and

$$f_k(s) = \frac{1}{b^k (k-1)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} [(s - jb)_+]^{k-1},$$

where f_k is the density for the distribution F_k. The above formula for $F_k(s)$ ws originally obtained by Laplace [33].

It should be noted however, that the evaluation of

$$\int [(1-s)_+]^{k-1} f_k(s)ds$$

becomes quite tedious for larger values of k. Alternatively, for $k \leq \frac{1}{b}$ one may use

$$P\{\tau > k\} = b^{-k} \int_0^b \int_0^b \cdots \int_0^b (1 - x_1 - x_2 - \cdots - x_k)^{k-1} dx_1 dx_2 \cdots dx_k.$$

As another example, suppose the arc lengths are distributed such that

$$L_i = \begin{cases} a & \text{w.p. } q \\ b & \text{w.p. } 1-q \end{cases}$$

where $a < b$. Then for $k \geq 2$,

$$P\{\tau > k\} = \sum_{j=0}^{k} ([1 - ja - (k-j)b]_+)^{k-1} \binom{k}{j} q^j (1-q)^{k-j}. \tag{1.34}$$

Now some comparisons will be made between various values of $P\{\tau > k\}$ and $E[\tau]$ arising from different distributions all having the same mean. Consider eleven distributions, $G_1, \cdots, G_{11}$, each having mean $\frac{1}{100}$. Let G_1 be the trivial distribution which assigns probability 1 to the outcome $\frac{1}{100}$. Let G_2 be the uniform distribution on $(\frac{1}{200}, \frac{3}{200}]$, and let G_3 be the uniform distribution over $(0, \frac{1}{50}]$. Let G_4 assign equal probability to each of the values $\frac{1}{200}$ and $\frac{3}{200}$, and let G_5 give equal probability to the outcomes $\frac{1}{1000}$ and $\frac{19}{1000}$. Let G_6 be such that if $L \sim G_6$ then $P\left\{L = \frac{1}{150}\right\} = 2/3$ and $P\left\{L = \frac{1}{60}\right\} = 1/3$, and let G_7 be such that if $L \sim G_7$ then $P\left\{L = \frac{1}{300}\right\} = \frac{1}{3}$ and $P\left\{L = \frac{1}{75}\right\} = \frac{2}{3}$. Each of these first seven distributions has a variance that is no greater than 8.1×10^{-5}.

The variances of the next four distributions increase from a value of 4.725×10^{-4} for G_8 to a value of 2.34×10^{-3} for G_{11}. G_8 is such that if $L \sim G_8$ then $P\left\{L = \frac{1}{1000}\right\} = \frac{105}{123}$ and $P\left\{L = \frac{1}{16}\right\} = \frac{18}{123}$, and G_9 is such that if $L \sim G_9$ then $P\left\{L = \frac{1}{2000}\right\} = \frac{230}{249}$ and $P\left\{L = \frac{1}{8}\right\} = \frac{19}{249}$. Let G_{10} be such that if $L \sim G_{10}$ then $P\left\{L = \frac{1}{4000}\right\} = \frac{460}{499}$ and

$P\left\{L = \frac{1}{8}\right\} = \frac{39}{499}$. Finally, let G_{11} be such that if $L \sim G_{11}$ then $P\left\{L = \frac{1}{4000}\right\} = \frac{320}{333}$ and $P\left\{L = \frac{1}{4}\right\} = \frac{13}{333}$.

Let τ_j denote the time of the first collision for the packing sequence of random arcs having lengths distributed according to G_j. Then for $j = 1, \cdots, 11$, $P\{\tau_j > 2\} = 0.98$. The following table gives the exact values of $P\{\tau_j > 3\}$ and $P\{\tau_j > 4\}$ for $j = 1, \cdots, 11$, as well as estimates of the $E[\tau_j]$ based on 100,000 computer simulation trials for each distribution. ($E[\tau_1]$ is exact.)

j	$P\{\tau_j > 3\}$	$P\{\tau_j > 4\}$	$E[\tau_j]$
1	0.940900	0.88474	9.622
2	0.940925	0.88483	9.636
3	0.941000	0.88512	9.687
4	0.940975	0.88502	9.694
5	0.941143	0.88567	9.823
6	0.940967	0.88499	9.685
7	0.940967	0.88499	9.680
8	0.942318	0.89010	11.05
9	0.944178	0.89686	13.82
10	0.944264	0.89718	14.23
11	0.947920	0.90954	21.60

It may be seen that the values of $E[\tau_j]$ for the six nonconstant distributions having small variances are not too much greater than $E[\tau_1]$. For the four distributions with large variances, the values of $E[\tau_j]$ increase as the variances increase, and they are considerably larger than $E[\tau_1]$. Also, it is apparent that $E[\tau]$ is not asymptotically proportional to $(P\{A_1 \wedge A_2\})^{-1/2}$, as was the case when all of the arcs had the same length.

When comparing arc length distributions having the same expectation, it is conjectured that $E[\tau]$ increases as the variance does. So with a large variance the arcs tend to avoid each other and thus cover the circle at a faster rate than arcs of smaller variance

do. This is consonant with the conjecture examined by Siegel [49] and Huffer [24] dealing with the coverage of the circle by random arcs. They conjecture that when comparing arc length distributions having the same expectation, that if one concentrates more mass near the expectation, then the corresponding coverage probability will be smaller. In other words, they claim that a smaller variance gives a slower rate of coverage. But this suggests that the arcs may tend to overlap each other more frequently, which would mean that their conjecture is somewhat consistent with the observation that $E[\tau]$ decreases as the variance does.

An Unusual Case

An interesting case of a two outcome distribution is where one of the outcomes equals zero, and the other outcome is such that $E[L_i] = m$. That is, suppose

$$L_i = \begin{cases} 0 & \text{w.p. } q \\ \frac{m}{1-q} & \text{w.p. } 1-q, \end{cases} \tag{1.35}$$

where $0 < m \leq 1-q$. A collision occurs whenever any two arcs of positive length overlap, or whenever a point arc of zero length is covered by an arc of positive length. An easy argument establishes that (1.34) remains true even for this degenerate case. The following theorem is a statement of this fact.

Theorem 1.7. Suppose arc lengths $L_1, L_2, \cdots$ are distributed according to (1.35). Then

$$P\{\tau > k\} = \sum_{j=0}^{k} \left(\left[1 - (k-j)\frac{m}{1-q} \right]_+ \right)^{k-1} \binom{k}{j} q^j (1-q)^{k-j}.$$

Proof. Let $H_{k,j}$ be the event that exactly j of the first k arcs have length zero. Then

$$P\{\tau > k\} = \sum_{j=0}^{k} P\{\tau > k \mid H_{k,j}\} \binom{k}{j} q^j (1-q)^{k-j}. \tag{1.36}$$

Now

$$P\{\tau > k \mid H_{k,0}\} = \left(\left[1 - k\left(\frac{m}{1-q}\right) \right]_+ \right)^{k-1}. \tag{1.37}$$

and

$$P\{\tau > k \mid H_{k,k}\} = 1. \tag{1.38}$$

For $0 < j < k$, $P\{\tau > k \mid H_{k,j}\}$ is just the probability that $(k-j)$ random arcs of length $\frac{m}{1-q}$ are disjoint and that j random points avoid the portion of the circle covered by the positive length arcs. Upon conditioning, it may be seen that

$$\begin{aligned} P\{\tau > k \mid H_{k,j}\} &= \left(\left[1-(k-j)\frac{m}{1-q}\right]_+\right)^j \left(\left[1-(k-j)\frac{m}{1-q}\right]_+\right)^{k-j-1} \\ &= \left(\left[1-(k-j)\frac{m}{1-q}\right]_+\right)^{k-1}. \end{aligned}$$

Noticing that this formula also holds for the cases given by (1.37) and (1.38), it follows from (1.36) that

$$P\{\tau > k\} = \sum_{j=0}^{k} \left(\left[1-(k-j)\frac{m}{1-q}\right]_+\right)^{k-1} \binom{k}{j} q^j (1-q)^{k-j}$$

(which is the same as (1.34) for this case). ∎

Now suppose that $q = (1-m)$, so that

$$L_i = \begin{cases} 0 & \text{w.p. } 1-m \\ 1 & \text{w.p. } m. \end{cases} \tag{1.39}$$

Clearly, for $k \geq 2$

$$\begin{aligned} P\{\tau > k\} &= P\{H_{k,k}\} \\ &= (1-m)^k. \end{aligned}$$

Hence

$$\begin{aligned} E[\tau] &= 2 + \sum_{k=2}^{\infty} (1-m)^k \\ &= m + \frac{1}{m}. \end{aligned}$$

Putting $m = 0.01$ yields

$$E[\tau] = 100.01,$$

which is larger than the value of $E[\tau]$ for each of the previously considered distributions having $E[L_i] = 0.01$.

Another Interpretation

Now suppose that a collision is considered to have occurred only if there exists an overlap of positive length. Then, unlike the previous interpretation, the arcs of length zero cannot participate in a collision.

The distribution of τ is easy to determine in this case. If the arc lengths are distributed according to (1.35), then for $k \geq 2$

$$\begin{aligned} P\{\tau > k\} &= \sum_{j=0}^{k-2} P\{\tau > k \mid H_{k,j}\} \binom{k}{j} q^j (1-q)^{k-j} \\ &= \sum_{j=0}^{k-2} \left(\left[1 - (k-j)\frac{m}{1-q} \right]_+ \right)^{k-j-1} \binom{k}{j} q^j (1-q)^{k-j}. \end{aligned}$$

If the lengths are given by (1.39), then for $k \geq 2$

$$\begin{aligned} P\{\tau > k\} &= P\{H_{k,k} \cup H_{k,k-1}\} \\ &= (1-m)^k + k(1-m)^{k-1} m. \end{aligned}$$

It follows that in this case

$$\begin{aligned} E[\tau] &= 2 + \sum_{k=2}^{\infty} [(1-m)^k + k(1-m)^{k-1} m] \\ &= \frac{2}{m}. \end{aligned}$$

Some Bounds

As mentioned previously, the evaluation of the formulas for $P\{\tau > k\}$ becomes difficult unless k is rather small. It is therefore convenient to establish upper and lower bounds for these probabilities.

For any choice of F, (1.33) provides a lower bound for $P\{\tau > k\}$. It then follows from the proof of Proposition 1.11 that for $k < m_F^{-2/3}$

$$P\{\tau > k\} > e^{-m_F k(k-1)} (1 - m_F^{2/3} k),$$

provided that $m_F \leq \frac{1}{8}$.

An upper bound for $P\{\tau > k\}$ is given by the following theorem.

Theorem 1.8. For a sequence of random arcs having lengths $L_1, L_2, \cdots \overset{\text{iid}}{\sim} F$,

$$P\{\tau > k\} \leq \left(\int e^{-(k-1)x} dF(x)\right)^k \tag{1.40}$$

for any choice of F.

Proof. If $0 \leq S_k < 1$ then

$$\begin{aligned}(1 - S_k)_+ &= (1 - S_k) \\ &\leq e^{-S_k}.\end{aligned} \tag{1.41}$$

This inequality is also true for $S_k \geq 1$ since e^{-S_k} is always positive. It then follows from (1.41) and the independence of the L_i that

$$\begin{aligned}P\{\tau > k\} &= E\{[(1 - S_k)_+]^{k-1}\} \\ &\leq E\{e^{-(k-1)S_k}\} \\ &= E\{e^{-(k-1)(L_1 + \cdots + L_k)}\} \\ &= \left(\int e^{-(k-1)x} dF(x)\right)^k. \qquad \blacksquare\end{aligned}$$

More Examples

The upper bound (1.40) is generally much easier to evaluate than (1.32). For instance, if

$$P\{L_i = a\} = P\{L_i = b\} = \frac{1}{2},$$

then the upper bound (1.40) becomes

$$\left(\frac{e^{-(k-1)a} + e^{-(k-1)b}}{2}\right)^k = \left[\cosh^k\left((k-1)\left(\frac{b-a}{2}\right)\right)\right] e^{-k(k-1)(\frac{b+a}{2})}. \tag{1.42}$$

If F is the uniform distribution on $(a, b]$, then

$$\begin{aligned}P\{\tau > k\} &\leq \left(\frac{1}{b-a}\int_a^b e^{-(k-1)x} dx\right)^k \\ &= \left(\frac{e^{-(k-1)a} - e^{-(k-1)b}}{(k-1)(b-a)}\right)^k.\end{aligned} \tag{1.43}$$

It may be shown that this upper bound is bounded from above by (1.42).

To consider a specific example, suppose that

$$P\left\{L_i = \frac{1}{200}\right\} = P\left\{L_i = \frac{3}{200}\right\} = \frac{1}{2}.$$

Then the lower bound given by (1.33) is

$$P\{\tau > k\} \geq \left[\left(1 - \frac{k}{100}\right)_+\right]^{k-1},$$

so that

$$\begin{aligned} E[\tau] &= 2 + \sum_{k=2}^{\infty} P\{\tau > k\} \\ &\geq 2 + \sum_{k=1}^{99} \left(1 - \frac{k}{100}\right)^{k-1} \\ &= 9.622. \end{aligned}$$

Recalling for this distribution (G_4 from before), that the estimate of $E[\tau]$ obtained from 100,000 simulation trials was 9.694, it is observed that the lower bound is reasonably close.

For this case, the upper bound given by (1.42) becomes

$$P\{\tau > k\} \leq \left(\frac{\exp\left[-\frac{k-1}{200}\right] + \exp\left[-\frac{3(k-1)}{200}\right]}{2}\right)^k.$$

Since $P\{\tau > k\} = 0$ for $k > 199$,

$$\begin{aligned} E[\tau] &\leq 2 + \sum_{k=2}^{199} \left(\frac{\exp\left[-\frac{k-1}{200}\right] + \exp\left[-\frac{3(k-1)}{200}\right]}{2}\right)^k \\ &= 9.945. \end{aligned}$$

This upper bound for $E[\tau]$ is not as close as the corresponding lower bound; however, it only overestimates by about 2.6%.

An examination of the case where F is the uniform distribution over $\left[\frac{1}{200}, \frac{3}{200}\right]$ again finds that the bounds produce good estimates. The upper bound (1.43) yields

$$\begin{aligned} E[\tau] &\leq 2 + \sum_{k=2}^{199} \left(\frac{\exp\left[-\frac{k-1}{200}\right] - \exp\left[-\frac{3(k-1)}{200}\right]}{(k-1)/100}\right) \\ &= 9.904, \end{aligned}$$

a value that overestimates the observed value, 9.636, by about 2.8%.

The lower bound in this case has the same value as did the lower bound in the previous case, 9.622. This occurs because the lower bound provided by (1.33) depends only on the distribution through its mean, which is the same for both cases. It will be seen in the next case considered that the lower bound sometimes performs very poorly.

If the arc lengths are distributed according to (1.39) then

$$P\{\tau > k\} \leq [(1-m) + me^{-(k-1)}]^k,$$

so that

$$E[\tau] \leq 2 + \sum_{k=2}^{\infty}[(1-m) + me^{-(k-1)}]^k.$$

Putting $m = 0.01$ yields

$$E[\tau] \leq 100.025$$

which is not too much greater than the exact result,

$$E[\tau] = 100.01.$$

Notice however that the corresponding lower bound, which is again equal to 9.622, is extremely poor for this choice of F.

Another Approach

An alternative upper bound for $P\{\tau > k\}$ may be produced by the method used in Section 2. For instance, suppose F is the uniform distribution on $[a, b]$, and let p denote $P\{A_r \wedge A_s\}$ $(r \neq s)$. Then

$$p = 1 - P\{\tau > 2\} = b + a.$$

$P\{A_r \wedge A_s, A_t \wedge A_u\}$ is equal to p^2 if r, s, t and u are all different. If $s = u$, this probability is given by

$$\begin{aligned}
\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b & P\{A_r \wedge A_s, A_t \wedge A_s \mid L_r = x, L_t = y, L_s = z\}\,dx\,dy\,dz \\
&= \frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b (x+z)(y+z)\,dx\,dy\,dz \\
&= \frac{13}{12}(a+b)^2 - \frac{ab}{3}.
\end{aligned}$$

which is less than $\frac{13}{12}p^2$

Now for $k \geq 3$,

$$\begin{aligned}
P\{\tau > k\} &= P\left[\bigcap_{1\leq i<j\leq k}\{A_i \text{ does not collide with } A_j\}\right] \\
&\leq 1 - \binom{k}{2}p + p^2\left[\binom{\binom{k}{2}}{2} + \frac{1}{12}3\binom{k}{3}\right] \\
&= 1 - \frac{k(k-1)}{2}p + p^2\frac{k(k-1)(k-2)}{24}(3k+4).
\end{aligned}$$

The last expression above is less than or equal to the right side of (1.7) whenever

$$\text{(1.44)} \qquad \frac{(7k^2 - 17k + 3)}{12}p + \frac{(k-1)^5}{24}p^2 \leq 1.$$

Noting that the left side of (1.44) is less than

$$\frac{7(k-1)^2}{12}p + \frac{(k-1)^5}{24}p^2,$$

it is clear that (1.44) is true for all $k \leq 1 + \min\left\{\sqrt{\frac{6}{7p}}, \left(\frac{12}{p^2}\right)^{1/5}\right\}$.

The key steps in the proof of part (i) of the following proposition have now been established. The remaining details of the proof are omitted. Parts (ii) and (iii) are proved in the same way as the analogous propositions in Section 2.

Proposition 1.14. Suppose F is the uniform distribution on $[a, b]$ where $0 \leq a \leq b \leq \frac{1}{2}$.

(i) Let $M = \min\left\{\sqrt{\frac{6}{7(a+b)}}, \left(\frac{12}{(a+b)^2}\right)^{1/5}\right\}$. Then

$$P\{\tau > k\} < \begin{cases} e^{-\frac{(k-1)^2}{2}(a+b)} & \text{for } 2 \leq k \leq M \\ e^{-\frac{M^2}{2}(a+b)} & \text{otherwise.} \end{cases}$$

(ii) For fixed $t > 0$,

$$P\left\{\sqrt{\frac{a+b}{2}}\tau > t\right\} \sim e^{-t^2} \quad \text{as} \quad (a+b) \downarrow 0.$$

(iii) For $0 < \alpha \leq \frac{2}{5}$,

$$P\{\tau > [\![(a+b)^{-\alpha}]\!]\} \sim \exp\left[-\frac{1}{2}(a+b)^{1-2\alpha}\right] \quad \text{as} \quad (a+b) \downarrow 0$$

Results similar to these may be obtained for the discrete circle of length n, discussed in Section 3, as well. For example, suppose F is the distribution that assigns equal probability to the first m positive integers, where $m \leq \frac{n}{2}$. Then

$$P\{L_i = j\} = \frac{1}{m} \quad (i = 1, 2, \cdots, j = 1, \cdots, m),$$

where L_i is the length of arc A_i, as before. The clockwise endpoints $C_1, C_2, \cdots$ are selected independently and uniformly from $S = \{1, 2, \cdots, n\}$.

Now

$$p = P\{A_r \wedge A_s\} = \frac{m}{n},$$

$$P\{A_r \wedge A_s, A_t \wedge A_u\} = p^2$$

if r, s, t and u are all different, and for r, s, and t all different

$$P\{A_r \wedge A_s, A_t \wedge A_s\} = \frac{13}{12}p^2 - \frac{1}{12n^2} < \frac{13}{12}p^2.$$

Noting the similarities between these probabilities and the analogous ones for the continuous case, it follows that if $(a + b)$ is replaced by $p = \frac{m}{n}$, then the results stated in the last proposition hold for the discrete case also. Similar results are attainable for choices of F other than the uniform distribution, in both the discrete and continuous cases. However, not every choice of F can be successfully handled in this manner.

1.7. Packing Sequences in Two-Dimensional Spaces

In Section 4 the distribution of τ was investigated for the packing sequence of placing random arcs of length a on a circle of unit circumference. In this section, several two-dimensional analogs of that packing sequence will be examined.

The first case to be considered will be that of placing random circular disks of area v on a two-dimensional unit torus (see Miles [40] for a description of the k-dimensional unit torus). Each disk has radius $\sqrt{\frac{v}{\pi}}$, so that two disks collide if and only if the euclidean distance between their centers is less than $2\sqrt{\frac{v}{\pi}}$. Formally

$$S = (0, 1] \times (0, 1]$$

and

$$\delta = 2\sqrt{\frac{v}{\pi}}.$$

The points

$$C_i = (C_{i1}, C_{i2}) \qquad (i = 1, 2, \cdots)$$

are the centers of the disks, and they are selected randomly from S by letting the C_{ij} be independent, uniform $(0, 1]$ random variables. The distance between two points is given by

$$\mu(C_i, C_j) = ([\mu^*(C_{i1}, C_{j1})]^2 + [\mu^*(C_{i2}, C_{j2})]^2)^{1/2}$$

where

$$\mu^*(x, y) = \begin{cases} |x - y| & \text{if } |x - y| \leq \frac{1}{2} \\ 1 - |x - y| & \text{otherwise.} \end{cases}$$

This packing sequence has equivalent points, and

$$p = 4v.$$

Unlike the case for arcs on a circle, the distribution of τ is not known precisely in this two-dimensional extension. Propositions 1.1 and 1.2 provide lower and upper bounds for $P\{\tau > k\}$; however, the upper bound is not sufficiently tight over a large enough range to yield an asymptotic expression for $E[\tau]$ as p tends to zero. A plausible, but not exact, argument will be used below to derive an approximation for $E[\tau]$.

For each disk center C_i, let the associated "target area" be given by

$$E_i = \{(x, y) \in S : \mu((x, y), C_i) < \delta\}.$$

Thus a disk with center C_j will collide with the disk having center C_i if and only if $C_j \in E_i$, or equivalently if and only if $C_i \in E_j$. Also let

$$A_i = \{(x, y) \in S : \mu((x, y), C_i) < \delta/2\}.$$

Then A_i represents the actual disk of area v surrounding the point C_i.

Now $\{\tau > k-1\}$ implies that

$$A_i \cap A_j = \phi \quad \forall \ 1 \le i < j < k;$$

however, note that the collection of sets $\{E_i\}_{i=1}^{k-1}$ need not be mutually disjoint. The event $\{\tau \le k-1\}$ implies that $A_i \cap A_j \neq \phi$ for some $1 \le i < j < k$. This means that for at least one pair (i,j), the area of $E_i \cap E_j$ is greater than the maximum allowable overlap of target areas when all of the A_i are disjoint. Now it is heuristically assumed that conditioned on $\{\tau \le k-1\}$ having occurred, the area of $\cup_{i=1}^{k-1} E_i$ is not on the average larger than it is for when $\{\tau > k-1\}$. That is, it does not seem unreasonable to assume that

$$P\{C_k \wedge C(k-1) \mid \tau \le k-1\} \le P\{C_k \wedge C(k-1) \mid \tau > k-1\}, \tag{1.45}$$

which would imply that

$$P\{C_k \wedge C(k-1)\} \le P\{C_k \wedge C(k-1) \mid \tau > k-1\},$$

or equivalently

$$P\{C_k \vee C(k-1) \mid \tau > k-1\} \le P\{C_k \vee C_{k-1}\}.$$

If the above is true, it then follows that

$$\begin{aligned} P\{\tau > k\} &= P\{C_k \vee C(k-1), \tau > k-1\} \\ &= P\{C_k \vee C(k-1) \mid \tau > k-1\} P\{\tau > k-1\} \\ &\le P\{C_k \vee C(k-1)\} P\{\tau > k-1\}, \end{aligned}$$

and induction yields

$$P\{\tau > k\} \le P\{C_k \vee C(k-1)\} P\{C_{k-1} \vee C_{k-2}\} \cdots P\{C_2 \vee C_1\} P\{\tau > 1\}.$$

Since $P\{\tau > 1\} = 1$ and

$$P\{C_m \vee C_{m-1}\} = (1-p)^{m-1} \qquad (m = 2, 3, \cdots),$$

it follows that

$$\begin{aligned} P\{\tau > k\} &\le (1-p)^{\sum_{m=1}^{k-1} m} \\ &= (1-p)^{\frac{(k-1)k}{2}} \\ &< [(1-p)^{1/p}]^{\frac{(k-1)^2}{2}p} \\ &< e^{-\frac{(k-1)^2}{2}p}. \end{aligned}$$

If this upper bound is indeed true, then the result below easily follows from Theorem 1.2.

Theorem 1.9. Assume (1.45). Then for the packing sequence of circular disks on the torus

$$E[\tau] \sim \sqrt{\frac{\pi}{2p}} \quad \text{as} \quad p \downarrow 0, \tag{1.46}$$

or equivalently

$$E[\tau] \sim \sqrt{\frac{\pi}{8v}} \quad \text{as} \quad v \downarrow 0.$$

For another analog, consider the random placement of spherical caps having surface area v on a sphere of unit surface area. Each cap has angular radius $\alpha = 2\sin^{-1}(\sqrt{v})$, and two caps collide whenever the great circle distance between their centers is less than $\delta = 2\alpha$. This packing sequence has equivalent points with $p = 4v(1-v)$. Similar to the previous analog, it follows that for small p an approximation of $E[\tau]$ is given by

$$\sqrt{\frac{\pi}{2p}}.$$

Thus for v not too large,

$$E[\tau] \approx \sqrt{\frac{\pi}{8v(1-v)}}.$$

It is interesting to compare the above result with the analogous one for random disks on the unit torus. For disks and spherical caps having the same area, the results imply that the expected proportion of the total surface area covered immediately prior to the initial collision is greater for the caps on the sphere than it is for the disks on the torus. Since $P\{\tau > 2\} = 1 - p$ for packing sequences having equivalent points, it also follows that $P\{\tau > 2\}$ is greater for spherical caps than for disks of the same area. Because it is difficult to obtain closed form expressions for $P\{\tau > k\}$ for caps on the sphere when

$k \geq 3$, it will not be proven that $P\{\tau > k\}$ is always greater for spherical caps than for corresponding disks; however, simulation results (not reported here) indicate that this may be the case.

A third two-dimensional analog to arcs on a circle is the placement of squares of area v on a two-dimensional torus such that the two sides of each square are aligned to be parallel to a pair of perpendicular axes on the torus.

For this extension, the space S and the centers $C_1, C_2, \cdots$ are exactly the same as they were for the case of disks on the torus. However, now

$$\mu(C_i, C_j) = \max\{\mu^*(C_{i1}, C_{j1}), \mu^*(C_{i2}, C_{j2})\}$$

where μ^* is the same as before, and

$$\delta = \sqrt{v}.$$

This packing sequence has equivalent points, and it seems reasonable to suppose that, for small p,

$$E[\tau] \approx \sqrt{\frac{\pi}{2p}},$$

as it was for the case of disks on the torus. For squares of area v,

$$p = 4v,$$

so that

$$E[\tau] \approx \sqrt{\frac{\pi}{8v}}.$$

Thus the expected time to the first collision for random squares of area v on a unit torus appears to be approximately equal to the expected time to the first collision for random disks of area v on a unit torus.

The following table displays, for both settings, some estimates of $E[\tau]$ obtained by computer simulation, as well as the corresponding values of the approximation formula for $E[\tau]$. $\widehat{E[\tau_{d,v}]}$ denotes the observed average value of τ for disks of area v, and $\widehat{E[\tau_{s,v}]}$ denotes the corresponding value for squares of area v.

A Monte Carlo study of expected time to first collision on the torus

$\tau_{d,v}$ is the collision time for random disks of area v

$\tau_{s,v}$ is the collision time for random squares of area v

v	# trials	$\widehat{E[\tau_{d,v}]}$	$\widehat{E[\tau_{s,v}]}$	$\sqrt{\frac{\pi}{8v}}$
$\frac{1}{16}$	60000	3.3255	3.3282	2.5066
$\frac{1}{25}$	50000	3.9517	3.9570	3.1333
$\frac{1}{36}$	50000	4.5833	4.5862	3.7599
$\frac{1}{64}$	50000	5.8134	5.8334	5.0133
$\frac{1}{81}$	50000	6.4412	6.4551	5.6399
$\frac{1}{100}$	110000	7.0696	7.0753	6.2666
6.4×10^{-3}	50000	8.6662	8.6608	7.8832
2.5×10^{-3}	50000	13.345	13.335	12.533
1.6×10^{-3}	50000	16.525	16.515	15.666
9.0×10^{-4}	50000	21.697	21.713	20.889
6.25×10^{-4}	50000	25.947	25.822	25.066
4.0×10^{-4}	10000	32.193	32.110	31.333
1.0×10^{-4}	10000	63.440	62.725	62.666
2.5×10^{-5}	10000	126.90	125.66	125.33
1.6×10^{-5}	10000	158.63	157.80	156.66
9.0×10^{-6}	10000	209.35	207.71	208.89
6.25×10^{-6}	10000	251.95	252.90	250.96
4.0×10^{-6}	10000	313.93	312.45	313.33
1.0×10^{-6}	5000	631.50	627.10	626.66

It may be seen above that for the larger values of v, $\widehat{E[\tau_{s,v}]}$ is greater than $\widehat{E[\tau_{d,v}]}$. For the smaller values of v considered, the reverse is true except in two cases. For these two cases, the apparent discrepancy may possibly be due to the smallness of the samples and the variability of τ.

It is interesting to note that for all $v \leq \frac{\pi}{64}$ the distribution of τ is different for disks and squares of the same area v. This may be established from the fact that

$$P\{\tau > 3\} = 1 - 12v + \left(32 + \frac{12\sqrt{3}}{\pi}\right) v^2$$

for disks of area $v \leq \frac{\pi}{64}$, and

$$P\{\tau > 3\} = 1 - 12v + 39v^2$$

for squares of area $v \leq \frac{1}{16}$. These probabilities were calculated explicitly.

Packing sequences may also be defined in spaces having dimension greater than two. For example, consider the three-dimensional extension of the last packing sequence described. Such a packing sequence corresponds to placing at random cubes of volume v in a three-dimensional unit torus. A collision occurs whenever two cubes overlap.

For $v \leq \frac{1}{8}$,

$$P\{\tau > 2\} = 1 - p = 1 - 8v.$$

A direct calculation yields that if $v \leq \frac{1}{64}$ then

$$P\{\tau > 3\} = 1 - 24v + 165v^2.$$

The computation of $P\{\tau > k\}$ for k greater than 3 is not easily accomplished; however, it is not unreasonable to expect that if v is sufficiently small then $P\{\tau > k\}$ is a polynomial in v of degree $k - 1$ having

$$1 - k(k-1)4v = 1 - \binom{k}{2} p$$

as its first terms.

An argument similar to the one given for the two-dimensional cases suggests that $E[\tau]$ should be closely approximated by

$$\sqrt{\frac{\pi}{2p}} = \sqrt{\frac{\pi}{16v}}.$$

To be more precise, if (1.45) is true then it easily follows that $E[\tau] \sim \sqrt{\frac{\pi}{16v}}$.

The table below contains estimates of $E[\tau]$ obtained by computer simulation, along with corresponding values obtained from the approximation formula above.

v	# trials	$\widehat{E[\tau]}$	$\sqrt{\frac{\pi}{16v}}$
$\frac{1}{64}$	10^4	4.372	3.545
$\frac{1}{512}$	10^4	10.891	10.027
$\frac{1}{1000}$	10^4	14.829	14.012
$\frac{1}{8000}$	10^4	40.004	39.633
$\frac{1}{512000}$	10^3	311.652	317.066
$\frac{1}{10^6}$	10^3	452.472	443.113
$\frac{1}{(8\times 10^6)}$	200	1229.625	1253.314

1.8. Random q-ary Codewords

Consider the metric space consisting of all q-ary n-tuples with Hamming distance as the metric. That is, let

$$S = \{(s_1, s_2, \cdots, s_n) : s_i \in \{0, \cdots, q-1\} \quad (i = 1, \cdots, n)\},$$

and for $x = (x_1, \cdots, x_n) \in S$ and $y = (y_1, \cdots, y_n) \in S$ let

$$\mu(x, y) = n - \sum_{j=1}^{n} \delta_0^{x_j - y_j},$$

where

$$\delta_0^w = \begin{cases} 1 & \text{if } w = 0 \\ 0 & \text{if } w \neq 0. \end{cases}$$

Thus the Hamming distance between two points is just the number of coordinates in which they differ.

Note that there are q^n points in the space. If two points collide whenever the Hamming distance between them is less than $d \in \{1, 2, \cdots\}$, then the probability that two arbitrary points collide is just

$$p(q, n, d) = \frac{1}{q^n} \sum_{j=0}^{d-1} \binom{n}{j} (q-1)^j.$$

Note that for d equal to 1, the setting is the same as that of the basic birthday problem when there are q^n days in a year.

For the case of q equal to 2 and $d \geq 1$ fixed, a result of Kozlov [32] yields that as n and k tend to infinity, if

$$\binom{k}{2} p(2, n, d) \to \lambda$$

then

$$P\{\tau > k\} \to \epsilon^{-\lambda}.$$

This suggests that the approximation

$$P\{\tau > k\} \approx \epsilon^{-\binom{k}{2} p(2,n,d)}$$

should not be too bad if n and k are not too small.

The results of Section 2 may also be applied in this setting, since the packing sequence has equivalent points. Proposition 1.3 yields that

$$P\left\{ \left(\frac{p(q,n,d)}{2} \right)^{1/2} \tau > t \right\} \sim \epsilon^{-t^2} \quad \text{as} \quad n \to \infty,$$

for fixed q and d. Furthermore, an argument similar to the one preceding (1.45), combined with the result of Theorem 1.1, suggests that the approximation

$$E[\tau] \approx \sqrt{\frac{\pi}{2p(q,n,d)}} \tag{1.47}$$

should do reasonably well provided that n is not too small.

As an example, consider the case where $q = 2$ and $d = 3$. Letting $\widehat{E[\tau]}_m$ denote the average value of τ obtained from m random packing sequences performed by a computer using a pseudo random number generator, the table below displays values of $\widehat{E[\tau]}_m$ and the approximation given by (1.47) for various values of n.

n	m	$\widehat{E[\tau]}_m$	$\sqrt{\frac{\tau}{2p(2,n,3)}}$
9	1000	4.96	4.18
11	1000	7.73	6.93
13	1000	12.52	11.83
15	500	21.70	20.62
17	500	36.92	36.56
19	500	66.51	65.66
21	250	119.92	119.16
23	250	217.41	218.10

The following table shows some corresponding results for the case of $q = 2$ and $d = 5$.

n	m	$\widehat{E[\tau]}_m$	$\sqrt{\frac{\tau}{2p(2,n,5)}}$
15	500	6.08	5.15
16	500	7.00	6.40
17	500	8.45	8.00
18	500	10.89	10.09
19	500	14.08	12.79
20	250	17.16	16.30
21	250	21.68	20.89
22	250	27.06	26.89
23	250	34.56	34.76
24	250	47.86	45.11

Now consider an alternative packing sequence on the space of ternary n-tuples having a metric μ given by

$$\mu(x, y) = \max_{1 \le i \le n} |x_i - y_i|.$$

Two random points collide whenever the distance between them is less than 2. Thus two points C_i and C_j do not collide if and only if for some m, $C_{im} = 0$ and $C_{jm} = 2$, or

$C_{im} = 2$ and $C_{jm} = 0$.

This packing sequence may also be described as follows. Imagine a cube in n dimensions of sidelength 4, having a cubic lattice of unit sidelength superimposed within it so that each vertex of the cube coincides with a lattice point. Smaller cubes, having sidelength 2, are then sequentially placed at random within the larger cube so that each vertex coincides with one of the lattice points. This may be accomplished by letting the C_{ij} $(i = 1, 2, \cdots, j = 1, \cdots, n)$ be i.i.d. random variables having a uniform distribution on the set $\{0, 1, 2\}$. Equivalently, each of the small cubes is put uniformly at random at one of 3^n possible locations within the big cube.

The centermost location within the large cube is labeled $(1, 1, \cdots, 1)$. The sides of a small cube placed at this location do not touch any of the sides of the large cube. This is not true if a small cube is placed at any other location.

A collision occurs whenever any two small cubes are not disjoint. Notice that a small cube placed at $(1, \cdots, 1)$ will collide with any other cube.

The probability that any two arbitrary points collide is

$$p = \left(\frac{7}{9}\right)^n .$$

This follows from the fact that

$$\begin{aligned}
&P\{|C_{im} - C_{jm}| < 2\} \\
&= P\{|C_{im} - C_{jm}| < 2 \mid C_{jm} = 1\} \left(\frac{1}{3}\right) + P\{|C_{im} - C_{jm}| < 2 \mid C_{jm} \in \{0, 2\}\} \left(\frac{2}{3}\right) \\
&= (1)\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) \\
&= \frac{7}{9}.
\end{aligned}$$

The packing sequence does not have equivalent points since

$$P\{C_i \wedge C_j \mid C_j = (1, \cdots, 1)\} = 1 \neq p.$$

Thus Theorem 1.1, which provides a lower bound for $E[\tau]$ which is asymptotically equivalent to $\sqrt{\frac{\pi}{2p}}$, is not applicable. The argument in Section 7 which suggests that $E[\tau]$ is

bounded from above by $\sqrt{\frac{\pi}{2p}}$ is also not applicable.

In order to see whether or not $\sqrt{\frac{\pi}{2p}}$ provides a good approximation for $E[\tau]$ for this packing sequence, random packings were generated by computer for various values of n. The table below displays results obtained from these packing attempts, along with the corresponding values of $\sqrt{\frac{\pi}{2p}}$. The notation is the same a before. Estimates of the standard deviation of $\widehat{E[\tau]}_m$ are also given since τ has great variability in this packing scheme. Although it is not conclusive, the results indicate that, for small p, $\sqrt{\frac{\pi}{2p}}$ approximates $E[\tau]$ reasonably well even in this unusual case.

n	m	$(\frac{9}{7})^{n/2}\sqrt{\frac{\pi}{2}}$	$\widehat{E[\tau]}_m$	s.d. $(\widehat{E[\tau]}_m)$
5	10000	2.35	3.37	0.01
10	10000	4.40	5.66	0.03
15	5000	8.25	9.80	0.07
20	2000	15.47	17.06	0.20
25	1000	29.00	32.00	0.53
30	1000	54.35	57.26	0.96
35	500	101.88	106.50	2.58
40	100	190.96	188.69	9.59
45	100	357.94	365.94	17.20
50	100	670.92	638.79	32.75

The table below compares the distribution of τ for packing sequences of arcs and simple cubes for which p is the same. The results for the arcs on the circle are exact, while the simple cubic results are estimated from the simulation trials.

		$E[\tau]$		s.d. (τ)	
p	$\sqrt{\frac{\pi}{2p}}$	arcs	cubes	arcs	cubes
6.56×10^{-3}	15.47	16.23	17.06	7.84	9.04
1.87×10^{-3}	29.00	29.75	32.00	14.91	16.67
5.32×10^{-4}	54.35	55.11	57.26	28.17	30.40

It may be seen above that $\sqrt{\frac{\pi}{2p}}$ underestimates $E[\tau]$ in all cases, but does so more severely for the cubes than for the arcs. Also notice that the standard deviation of τ is greater for the cubes than it is for the arcs.

1.9. Summary

It has been shown that the approximation

$$E[\tau] \approx \sqrt{\frac{\pi}{2p}} \tag{1.48}$$

holds in a wide variety of collision settings. For cases considered for which the distribution of τ is known exactly, the two sides of (1.48) are proven to be asymptotically equivalent. For packing sequences of equivalent points, asymptotic equivalence is made plausible but has not been proven. That (1.48) can hold for a packing sequence not possessing equivalent points was demonstrated by the simple cubic packing scheme. The formula (1.48) does not hold for arcs of unequal or variable length on the circle: however, note that in these settings a collision is no longer defined by the simple relation

$$C_i \wedge C_j \Leftrightarrow \mu(C_i, C_j) < \delta$$

for some fixed δ.

The approximation formula

$$P\{\tau > k\} \approx e^{-\frac{k^2}{2}p} \tag{1.49}$$

has also been investigated. For random arcs of constant length, (1.49) is reasonably accurate whenever $k < p^{-2/3}$. This is suggested by Proposition 1.12 and demonstrated by

the results shown in Table A. The results of Table A also indicate that (1.49) can be quite inaccurate if $k > p^{-2/3}$. It is reasonable to suppose that the cutoff point $p^{-2/3}$ occurs in other packing schemes as well.

Chapter 2

Approximate Packing Densities of Randomly Constructed Codes

2.1. Introduction

Consider the sequence of points $C_1, C_2, \cdots$ chosen independently and uniformly from S. Each point in the sequence will be considered to be either packed or rejected. A point will be rejected if and only if it collides with a previously packed point. Otherwise the point will be packed. Thus C_1 will be packed, and C_2 will be packed unless it collides with C_1. Then, letting $\mathcal{D}_i$ denote the union of all points from among $C_1, \cdots, C_i$ which have been packed, C_{i+1} will be packed unless it collides with some member of $\mathcal{D}_i$.

The members of $\mathcal{D}_i$ are said to constitute a saturated packing if each point in S collides with at least one element of $\mathcal{D}_i$. Thus, if $\mathcal{D}_i$ is a saturated packing, none of $C_{i+1}, C_{i+2}, \cdots$ can be packed and $\mathcal{D}_i = \mathcal{D}_{i+1} = \mathcal{D}_{i+2} = \cdots$.

Let T be defined as the first time i for which $\mathcal{D}_i$ is a saturated packing. Call T, a stopping time, the time to saturation for the packing sequence. Let M be the random variable defined by

$$M = \#\mathcal{D}_T.$$

Thus M counts the number of points in a saturated packing, and T is the total number of selections from S, including rejections, needed to reach saturation.

For a metric space $\langle S, \mu\rangle$, suppose a suitably defined content v is associated with each point. Then let the random packing density, denoted by ρ, be defined as

$$\rho = vE[M].$$

If S is finite, the content v may be taken to be $(\#S)^{-1}$. For this choice of v, the packing density will be referred to as a center density and will be denoted by r. Hence

$$r = \frac{1}{\#S} E[M].$$

Random packing problems have been studied by numerous investigators; however, attempts to obtain exact solutions have been met with very little success except for the cases of packing on the discrete and continuous circles (or line segments). The one-dimensional variations are often referred to as "parking problems" since the general idea can be expressed by the question 'How many cars of length a can eventually be parked on a street of length x if the parking is done at random?'. Put this way, it is also assumed that parked cars are never moved and that cars will move on to other streets if they cannot fit next to the curb at their randomly chosen locations.

Random packing problems in spaces of dimensions two and three are also of interest, partially because they can be used to model physical phenomena such as molecular adsorption and liquid and plasma structure. It is also of interest to compare the packing densities obtained with the density from an analogous one-dimensional case. A conjecture of Palásti [44] suggests that a two-dimensional density should be equal to the associated one-dimensional density squared; however, the conjecture has never been proven and numerous simulation studies have indicated that it is not true. More will be said about this conjecture in Section 2.5.

Solomon [53] reviews the major findings in the one-dimensional settings, and discusses efforts to approximate densities in higher dimensions by simulation results. A more recent survey article by Solomon and Weiner [55] updates this material.

The study of random packing densities in multidimensional spaces through computer generated packings is, in general, difficult due to the extremely long running times required to perform the random packings. However, if each coordinate of a random point must assume one of only a small number of possible values, then the time required to achieve a saturated packing need not be unreasonably long. Random packing sequences on the space

of q-ary codewords provide convenient and interesting ways to study packing densities in spaces of two or more dimensions. Various random coding schemes will be examined in the next five sections.

2.2. Random Binary Codes

Consider the metric space introduced in Section 8 of Chapter 1, and let q equal 2. Thus S is the space of all binary codewords of length n. There are 2^n such codewords, and each one can be represented by a unique n-component vector with each component being either 0 or 1.

The Hamming weight of a codeword $\mathbf{u}$ is defined to be the number of nonzero components of $\mathbf{u}$, and the Hamming distance between two codewords $\mathbf{u}$ and $\mathbf{v}$ is the Hamming weight of $\mathbf{u} - \mathbf{v}$, where modulo-2 arithmetic is applied. Note that this definition of Hamming distance is equivalent to the one given in Chapter 1.

Subsets of S containing two or more elements will be called codes. A code is called an (n,d)-code if the codewords are of length n and the distance between each pair of words is greater than or equal to d. The minimum distance d is an important parameter in the description of a code since it is related to the error detecting and error correcting capabilities of the code. It is possible to detect up to $d-1$ errors, where an error is said to occur when a bit is recorded incorrectly at the receiving end. Furthermore, up to $[\![(d-1)/2]\!]$ transmission errors can always be successfully corrected.

Usually codes are constructed by algebraic methods in order to "neatly arrange" the codewords, and hopefully produce (n,d)-codes which have the maximum number of words. $A(n,d)$ will denote the number of words in the largest possible (n,d)-code. Codes constructed algebraically may also have nice properties which make decoding and error correcting easier. Despite a large literature, the known "good" codes are relatively few in number.

The problem investigated in this section is the random generation of various types of binary error-correcting codes. While the primary motivation for this work was to learn

something about random packing densities in high-dimensional spaces, it is also of interest to see how the sizes of random codes compare with the sizes of similar codes constructed deterministically.

Itoh and Solomon [27] consider the random sequential construction of binary codes for various values of n and d. They begin the construction process by selecting at random a single codeword from S. Then the second word is chosen at random from among all codewords which are at Hamming distance d or greater from the initial word. This procedure is continued, at each step choosing a new codeword from the collection of all codewords which are at a Hamming distance of d or greater from all of the codewords previously selected. The process terminates when it is no longer possible to add another codeword to the chosen set. This procedure corresponds to the sequential packing scheme described in Section 1, provided that a pair of codewords are considered to collide whenever the Hamming distance between them is less than d. Thus the random (n, d)-code formed is just a saturated packing of codewords.

Letting $M(n, d)$ denote the number of words contained in a saturated packing, the center density is denoted

$$r_{d,n} = 2^{-n} E[M(n, d)].$$

This density may be estimated by

$$\hat{r}_{d,n} = 2^{-n} \overline{M(n, d)},$$

where $\overline{M(n, d)}$ is the average number of words packed in a Monte Carlo experiment. That is, if N random packing attempts result in codes being formed having $X_1, X_2, \cdots, X_N$ words, then $\overline{M(n, d)} = \frac{1}{N} \sum_{i=1}^{N} X_i$.

The results of numerous packing attempts performed by computer are summarized in Appendix A. The data for binary cases with $3 \leq n \leq 17$ are taken from [27]. The data for cases with $n > 17$ are results from more recent simulations. It should be noted that the number of trials per case ranges from 10,000 for cases having $n \leq 10$ to only 10 trials for most cases having $n \geq 17$.

For example, consider the first entry in Appendix A. This says $q = 2$ (a binary problem), $n = 3$ (so binary triples are being considered), and $d = 2$ (so any two words in the packing must differ in at least two coordinates). The Monte Carlo estimates for the mean and the standard deviation are about 3.49 and 0.87. Unfortunately, large standard deviations are an inherent part of the problem in this case as well as in most of the other cases considered.

Itoh and Solomon propose that the form $n^{-\gamma_d}$ approximates the $\hat{r}_{d,n}$ reasonably well for the cases $d = 2$ and $d = 3$, where the γ_d are constants. They estimate γ_2 and γ_3 from logarithmic plots of the simulation results using the least squares method. Only the cases for which $10 \leq n \leq 17$ are used in the fit.

The following tables compare their fitted values with the simulation results. Below, and throughout the chapter, s.d. will be used to denote the estimated standard deviation obtained from the sample.

$d = 2$ cases

$\gamma_2 \doteq 0.6249$

n	$\hat{r}_{2,n}$	$n^{-\alpha_2}$	$(n^{-\alpha_2} - \hat{r}_{2,n})/\text{s.d.}(\hat{r}_{2,n})$
8	0.26914	0.27269	12.7
9	0.25215	0.25334	6.1
10	0.23656	0.23720	5.0
11	0.22324	0.22349	0.9
12	0.21179	0.21166	-0.7
13	0.20187	0.20133	-1.4
14	0.19257	0.19222	-1.5
15	0.18399	0.18411	0.7
16	0.17677	0.17683	0.5
17	0.17018	0.17026	0.3

$d = 3$ cases

$\gamma_3 \doteq 1.319$

n	$\hat{r}_{3,n}$	$n^{-\alpha_3}$	$(n^{-\alpha_3} - \hat{r}_{3,n})/\text{s.d.}(\hat{r}_{3,n})$
8	0.06430	0.06435	1.5
9	0.05543	0.05509	-13.3
10	0.04834	0.04794	-23.2
11	0.04263	0.04228	-9.9
12	0.03800	0.03769	-12.9
13	0.03410	0.03392	-3.1
14	0.03077	0.03076	-0.3
15	0.02802	0.02808	2.9
16	0.02557	0.02579	14.1
17	0.02346	0.02381	8.5

It can be seen that the fit is rather good for $d = 2$ and $n \geq 11$; however, the model does not provide a very good fit for the case of $d = 3$.

In Chapter 1 it was shown that the expected value of the stopping time τ could be approximated by a simple function of p, the probability that two arbitrary points collide. Similarly, for binary codeword packing, the center densities $r_{d,n}$ can be closely approximated by a function of the ratio $\frac{p}{v}$ provided that n is large enough. For the space of binary codewords of length n, the content v is denoted by v_n and

$$v_n = (\# S)^{-1} = 2^{-n}.$$

Let $p_{d,n}$ be the probability that two randomly selected codewords of length n are separated by a Hamming distance less than d. Then

$$p_{d,n} = 2^{-n} \sum_{j=0}^{d-1} \binom{n}{j}.$$

Letting $\theta_{d,n}$ denote the ratio $p_{d,n}/v_{d,n}$, it follows that

$$\theta_{d,n} = \sum_{j=0}^{d-1} \binom{n}{j}.$$

It is found that

$$f_d(\theta_{d,n}) = \theta_{d,n}^{-\beta_d} + \alpha_d \theta_{d,n}^{-2\beta_d}$$

approximates the observed $\hat{r}_{d,n}$ rather well for n not too small, where the α_d and the β_d are constants. The values of these constants are determined by performing a weighted nonlinear least squares fit on each set of points $\hat{r}_{d,L(d)}, \hat{r}_{d,L(d)+1}, \cdots, \hat{r}_{d,U(d)}$ $(d = 1, \cdots, 10)$ using a modified Gauss-Newton algorithm (BMDP program 3R was used). Each case was weighted by the inverse of the estimated variance of $\hat{r}_{d,n}$. The $L(d)$ were determined by trial and error, and were chosen to be as small as possible while keeping the resulting fit acceptably accurate. The $U(d)$ are chosen to be as large as possible, the values being limited only by the enormous amount of computer time required to perform the simulations.

The following table summarizes the parameters resulting from the curve fittings.

Parameters determined for binary cases

d	$L(d)$	$U(d)$	$\hat{\alpha}_d$	$\hat{\beta}_d$
2	9	17	0.331	0.630
3	12	17	-0.870	0.740
4	10	17	-3.20	0.736
5	12	18	-22.1	0.795
6	11	18	-55.6	0.795
7	15	18	-2.04×10^2	0.833
8	14	18	-7.60×10^2	0.831
9	17	19	-4.48×10^3	0.851
10	17	20	-7.68×10^3	0.861

The entries in the next table are the residuals, $\hat{f}_d(\theta_{d,n}) - \hat{r}_{d,n}$, divided by the estimated standard deviation of $\hat{r}_{d,n}$.

Standardized residuals

n	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$	$d=10$
9	0.8	11.9	-6.0	-29.2	–	–	–	–	–
10	0.3	11.3	-0.1	-37.5	-39.3	–	–	–	–
11	-0.3	3.0	0.8	-8.7	0.1	-65.3	–	–	–
12	-1.6	0.3	-0.8	-0.3	-0.2	42.2	-94.8	–	–
13	-1.6	-0.4	0.6	0.9	-0.2	6.4	-46.7	–	–
14	-1.5	0.5	0.2	0.1	0.7	4.7	-0.5	–	–
15	0.9	-1.9	-0.8	0.4	0.0	0.0	0.9	–	–
16	0.9	0.6	0.4	0.1	0.2	-0.2	0.1	-16.2	-52.7
17	0.4	1.2	0.6	-1.4	0.3	1.3	-0.4	-0.9	-0.3
18	–	–	–	0.4	-1.6	-0.7	-0.8	1.1	-0.1
19	–	–	–	–	–	–	–	-0.6	2.1
20	–	–	–	–	–	–	–	–	-1.3

For example, consider the $d = 2$ entry of the first table. This shows that the fit was done using $\hat{r}_{2,9}, \hat{r}_{2,10}, \cdots, \hat{r}_{2,17}$, and that these points led to the estimates of 0.331 and 0.630 for α_2 and β_2 respectively. The standardized residuals from this fit are given in the first column of the second table. For instance, the $n = 11$ entry shows that $\hat{f}_2(\theta_{2,11})$ under approximates $\hat{r}_{2,11}$ by 0.3 s.d.$(\hat{r}_{2,11})$. Here

$$\hat{f}_2(\theta_{2,11}) = \theta_{2,11}^{-\hat{\beta}_2} + \hat{\alpha}_2\theta_{2,11}^{-2\hat{\beta}_2}.$$

Note from the first table above that $\beta_3 \approx \beta_4$, $\beta_5 \approx \beta_6$, $\beta_7 \approx \beta_8$, and $\beta_9 \approx \beta_{10}$. Further note that if $B(d) = \lceil \frac{d+2}{2} \rceil$, then $\beta_k^{B(k)}$ is approximately constant. This fact will be discussed further.

If the $\hat{r}_{d,n}$ are fit with the approximation

$$g_d(\theta_{d,n}) = \theta_{d,n}^{-\beta^{\frac{1}{B(d)}}} + \alpha_d\theta_{d,n}^{-2\beta^{\frac{1}{B(d)}}},$$

the estimated $\hat{\alpha}_d$ remain the same as before. The estimate of β gotten from each fit is shown below, along with corresponding estimates of the standard deviation of $\hat{\beta}$. The estimated standard deviations are computed with a provisional means algorithm (see [10]).

d	$\hat{\beta}$	s.d. $(\hat{\beta})$
2	0.3975	0.0009
3	0.4059	0.0006
4	0.3990	0.0009
5	0.3985	0.0004
6	0.3995	0.0013
7	0.4022	0.0030
8	0.3962	0.0038
9	0.3800	0.0126
10	0.4074	0.0148

These results suggest that perhaps

$$r_{d,n} \sim \theta_{d,n}^{-\beta^{\frac{1}{B(d)}}} \quad \text{as} \quad n \to \infty,$$

for some β approximately equal to 0.40. This asymptotic form has the ratio $r_{d,n+1}/r_{d,n}$ tending to 1 as n tends to infinity, in agreement with an observation made by Itoh and Solomon [27]. As with other packing settings, there does not seem to be a simple heuristic argument which suggests the limiting form of the densities.

A Comparison with the Hamming Bound.

The Hamming, or sphere-packing, bound for codes guarantees that any t-error-correcting binary code of length n containing M codewords satisfy

$$M\left[1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{t}\right] \leq 2^n$$

(see [38]). Since the number of bit errors an (n,d)-binary code can successfully correct is

$$[(d-1)/2],$$

the bound implies that the number of words in any (n, d)-code cannot exceed

$$2^n \left[1 + \binom{n}{1} + \cdots + \binom{n}{(\frac{d-1}{2})}\right]^{-1}$$

if d is odd, and the number of words cannot exceed

$$2^n \left[1 + \binom{n}{1} + \cdots + \binom{n}{(\frac{d-2}{2})}\right]^{-1}$$

if d is even.

It is also true that any saturated packing which constitutes an (n, d)-binary code must contain at least

$$2^n \left[1 + \binom{n}{1} + \cdots + \binom{n}{d-1}\right]^{-1}$$

words. This follows from the fact that the addition of each new word to the packed collection can decrease the number of available sites by at most $\sum_{j=0}^{d-1} \binom{n}{j}$.

The above bounds can be used to establish the facts that

$$r_{d,n} \geq L(d, n) = \left[1 + \binom{n}{1} + \cdots + \binom{n}{d-1}\right]^{-1},$$

and

$$r_{d,n} \leq U(d, n) = \begin{cases} \left[1 + \binom{n}{1} + \cdots + \binom{n}{(\frac{d-1}{2})}\right]^{-1} & (d \text{ odd}) \\ \left[1 + \binom{n}{1} + \cdots + \binom{n}{(\frac{d-2}{2})}\right]^{-1} & (d \text{ even}). \end{cases}$$

Table B contains the ratios $\hat{r}_{d,n}/L(d, n)$, and Table C contains the ratios $\hat{r}_{d,n}/U(d, n)$. From an inspection of these tables it can be seen that if d is held fixed and n is made larger, then the ratios of form $\hat{r}_{d,n}/L(d, n)$ tend to increase with n, while the ratios of the form $\hat{r}_{d,n}/U(d, n)$ are decreasing as n increases.

Now if the proposed asymptotic form for $r_{d,n}$ given by

$$R(d, n) = \theta_{d,n}^{-\beta^{\frac{1}{B(d)}}}$$

is correct, then as n becomes large

$$R(d, n)/L(d, n)$$

Table B

The entries of this table are the ratios $\hat{r}_{d,n}/L(d,n)$

n	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$	$d=10$
3	1.75	–	–	–	–	–	–	–	–
4	1.94	–	–	–	–	–	–	–	–
5	2.08	1.94	–	–	–	–	–	–	–
6	2.21	2.14	2.21	–	–	–	–	–	–
7	2.32	2.25	2.57	–	–	–	–	–	–
8	2.42	2.38	2.99	2.43	–	–	–	–	–
9	2.52	2.55	3.32	2.43	2.45	–	–	–	–
10	2.60	2.71	3.63	2.86	2.76	–	–	–	–
11	2.68	2.86	3.95	3.20	3.21	2.74	–	–	–
12	2.75	3.00	4.30	3.51	3.82	2.44	2.67	–	–
13	2.83	3.14	4.56	3.84	4.37	3.03	2.81	–	–
14	2.89	3.26	4.88	4.17	4.82	3.58	3.05	2.88	–
15	2.94	3.39	5.22	4.48	5.35	4.19	3.86	2.79	2.72
16	3.01	3.50	5.46	4.78	5.81	4.62	4.68	3.17	3.03
17	3.06	3.61	5.70	5.09	6.21	4.93	5.44	3.50	2.88
18	–	–	–	5.34	7.10	5.44	6.18	4.19	3.91
19	–	–	–	–	–	–	–	5.12	4.25
20	–	–	–	–	–	–	–	–	5.68

Table C

The entries in this table are the ratios $\hat{r}_{d,n}/U(d,n)$

n	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$	$d=10$
3	0.437	–	–	–	–	–	–	–	–
4	0.388	–	–	–	–	–	–	–	–
5	0.346	0.727	–	–	–	–	–	–	–
6	0.315	0.680	0.368	–	–	–	–	–	–
7	0.290	0.621	0.321	–	–	–	–	–	–
8	0.269	0.579	0.290	0.551	–	–	–	–	–
9	0.252	0.554	0.256	0.437	0.295	–	–	–	–
10	0.237	0.532	0.227	0.415	0.242	–	–	–	–
11	0.223	0.512	0.204	0.382	0.210	0.428	–	–	–
12	0.212	0.494	0.187	0.350	0.190	0.291	0.242	–	–
13	0.202	0.477	0.169	0.323	0.169	0.280	0.183	–	–
14	0.193	0.461	0.156	0.301	0.147	0.260	0.145	0.329	–
15	0.184	0.448	0.145	0.279	0.131	0.243	0.136	0.237	0.190
16	0.177	0.435	0.133	0.260	0.116	0.216	0.124	0.204	0.151
17	0.170	0.422	0.123	0.244	0.102	0.189	0.110	0.172	0.103
18	–	–	–	0.227	0.097	0.172	0.097	0.159	0.102
19	–	–	–	–	–	–	–	0.152	0.082
20	–	–	–	–	–	–	–	–	0.082

should tend to something greater than or equal to 1. Similarly,

$$R(d,n)/U(d,n)$$

should approach something less than or equal to 1 as n becomes large. It will now be shown that the first ratio tends to infinity, and that the second ratio has a limit of zero.

Note that $L(d,n)$ is equal to $\theta_{d,n}^{-1}$, and so

$$\begin{aligned}\lim_n \frac{R(d,n)}{L(d,n)} &= \lim_n \frac{\theta_{d,n}^{-\beta^{\frac{1}{B(d)}}}}{\theta_{d,n}^{-1}} \\ &= \lim_n \theta_{d,n}^{\{1-\beta^{\frac{1}{B(d)}}\}}.\end{aligned}$$

Since

$$\theta_{d,n} = O(n^{d-1}),$$

and since

$$1-\beta^{\frac{1}{B(d)}} > 0$$

if β is about 0.40, it follows that

$$\lim_{n\to\infty} \frac{R(d,n)}{L(d,n)} = \infty.$$

If d is even then

$$U(d,n) = \theta_{\frac{d}{2},n}^{-1},$$

and it follows that

$$\frac{R(d,n)}{U(d,n)} = O(n^{X(d)})$$

where

$$X(d) = \frac{d}{2} - 1 - (d-1)\beta^{\frac{1}{B(d)}}.$$

Now putting $\frac{2}{5}$ in for β and replacing $B(d)$ by $\frac{d+2}{2}$ yields

$$X(d) = \frac{d-2}{2} - (d-1)\left(\frac{2}{5}\right)^{\frac{2}{d+2}},$$

which is less than zero for all even values of d greater than or equal to two. Hence it now follows that

$$\lim_{n\to\infty} \frac{R(d,n)}{U(d,n)} = 0$$

for even values of d. It may similarly be shown that the above result is also true for the case where d is odd.

The above results indicate that the proposed asymptotic formula, $R(d,n)$, is consistent with the Hamming bounds. Another well known bound on the size of a code, the Plotkin bound, cannot be treated in a similar fashion. This is because the Plotkin bound only applies to cases for which $n < 2d$. Hence it is not possible to discuss its behavior when d is fixed and n is allowed to become large. It is also not very meaningful to compare the observed sizes of the random codes with the values of the maximum code size $A(n,d)$. This is due to the fact that the values of $A(n,d)$ are not known in all cases, and also because the values which are known tend to exhibit somewhat haphazard patterns.

2.3. Nonbinary codes

Nonbinary codes may also be sequentially constructed using the method described in the previous section. If m is a prime number, and q is any power of m, then a code with symbols from the Galois field $GF(q)$ is called a q-ary code. An (n,d)-q-ary code is a subset of n-tuples over $GF(q)$. The n-tuples, or codewords, are such that the Hamming distance between any pair of them is greater than or equal to d. As before, the Hamming distance between two words $\mathbf{u}$ and $\mathbf{v}$ is just the Hamming weight of $\mathbf{u} - \mathbf{v}$.

Ternary, or 3-ary, (n,d)-codes were stochastically formed by computer for $2 \le d \le 7$. As was the case for the binary codes, it is found that the observed $\hat{r}_{d,n}$ can be approximated by expressions of the form

$$f_d(\theta_{d,n}) = \theta_{d,n}^{-\beta_d} + \alpha_d \theta_{d,n}^{-2\beta_d}.$$

Note that for q-ary codeword packing the ratio p/v is given by

$$\theta_{d,n} = \sum_{j=0}^{d-1} \binom{n}{j} (q-1)^j.$$

The results of the random packings are summarized in the two tables below. The notation is the same as before, and additional details concerning the simulation results may be found in Appendix A.

Parameters determined for ternary cases

d	$L(d)$	$U(d)$	$\hat{\alpha}_d$	$\hat{\beta}_d$
2	5	9	-4.53×10^{-2}	0.632
3	7	9	-1.52	0.742
4	6	10	-10.4	0.764
5	8	10	-33.5	0.803
6	9	11	-2.32×10^{2}	0.819
7	10	11	-1.20×10^{3}	0.836

Standardized residuals

n	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
5	0.6	6.0	3.8	-	-	-
6	-0.6	6.2	-0.4	-	-	-
7	-0.3	-0.3	0.4	14.8	-	-
8	0.0	1.4	0.5	-0.2	-23.2	-
9	0.4	-0.5	-0.8	0.3	-0.1	-43.9
10	-	-	2.4	-1.4	1.0	0.0
11	-	-	-	-	-0.4	0.0

It is interesting to note that the values $\hat{\beta}_2$ and $\hat{\beta}_3$ are very close to the corresponding values for the binary case. It may also be noted that if

$$B(d) = \begin{cases} 2 & d = 2 \\ \frac{d+3}{2} & d > 2 \end{cases}$$

then the values $\hat{\beta}_d^{B(d)}$ $(d = 2, \cdots, 7)$ are approximately equal to one another, and therefore they are all near some unknown constant β.

Estimates of β from the ternary cases

d	$\hat{\beta}_d^{B(d)}$
2	0.399
3	0.408
4	0.390
5	0.416
6	0.408
7	0.407

Similar to the binary case, these results suggest that

$$r_{d,n} \sim \theta_{d,n}^{-\beta^{\frac{1}{B(d)}}} \quad \text{as} \quad n \to \infty,$$

for some β approximately equal to 0.40 or 0.41.

It is also interesting to compare the pattern of the values of $\mathrm{Var}(M(n,d))$ in the ternary case with the pattern found in the binary case. For the binary case, Itoh and Solomon [27] observed that the variance of packing density is larger when d is even and smaller when d is odd. Also it may be seen from Appendix A that the ratio

$$\mathrm{Var}(M(n,d))/\mathrm{Var}(M(n,d+1))$$

is generally greater than one when d is even, and generally less than one when d is odd. This pattern is not evident for the ternary data, as it may be seen from the results given in Appendix A that the above ratio is greater than one for all d. For the $q = 4$ and $q = 5$ cases, the values of $M(n,d)$ also strictly decrease as d increases (with n held fixed); however, it should be mentioned that there is insufficient data in these cases.

For $q > 3$, packings could be generated for only small values of n due to limitations on computer time. However, for each combination of q and d the values of $-\log \hat{r}_{d,n}/\log \theta_{d,n}$ suggest that perhaps

$$r_{d,n} \sim \theta_{d,n}^{-\beta_d} \quad \text{as} \quad n \to \infty$$

holds for $q = 4$ and $q = 5$. The tables below give the observed values of $-\log \hat{r}_{d,n} / \log \theta_{d,n}$ for $q = 4$ and $q = 5$. Additional simulation results may be found in Appendix A.

Estimates of the β_d from the 4-ary case

$q = 4$

n	$d = 2$	$d = 3$	$d = 4$
3	0.679	–	–
4	0.664	0.748	–
5	0.657	0.754	0.812
6	0.654	0.754	0.798
7	0.654	0.752	0.789

Estimates of the β_d from the 5-ary cases

$q = 5$

n	$d = 2$	$d = 3$	$d = 4$
3	0.695	–	–
4	0.679	0.758	–
5	0.671	0.753	0.803
6	0.667	0.754	0.800

2.4. Packing by Lee distance

The Lee distance between two q-ary codewords $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$ is given by

$$\mu(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \nu(x_i, y_i)$$

where

$$\nu(a, b) = \begin{cases} |a - b| & \text{if } |a - b| \leq q/2 \\ q - |a - b| & \text{otherwise.} \end{cases}$$

Two codewords are considered to collide if the Lee distance between them is less than some fixed $c \in \{1, 2, \cdots\}$.

Note that Lee distance is the same as Hamming distance if q is equal to 2 or 3. But for larger q, Lee distance is a more sensitive metric since the contribution from each

coordinate can be something other than just 0 or 1. Lee distance is sometimes used by coding theorists because it is well suited to phase-modulation. (The Hamming metric is well suited to orthogonal modulation schemes.)

Consider first the case where n is equal to 1. Then the metric space employing the Lee metric is identical to the one described in Section 3 of Chapter 1. A random packing on this metric space may be viewed as placing at random arcs of length c on a circle of circumference q, with the endpoints of the arcs being situated at integral coordinates. Nonoverlapping arcs are packed until the length of the longest segment of the circle not covered by an arc is less than c. Thus, this somewhat degenerate codeword packing problem is just a variation of a discrete, one-dimensional parking problem.

Let $M_{1,c,q}$ denote the number of arcs in the saturated packing, and let the content v of each packed point be $\frac{c}{q}$, which is just the proportion of the circle covered by each arc. Then the random packing density is given by

$$\rho_{1,c,q} = \frac{c}{q} E[M_{1,c,q}],$$

which is just the average proportion of the circle covered by a saturated packing of arcs. Note that $\rho_{1,1,q}$ is trivially equal to 1, since eventually every segment on the circle will be covered by a packed arc.

Page [43] and Downton [12] have studied this packing sequence for the special case of c equal to 2. They proved that as q tends to infinity, $p_{1,2,q}$ tends to the finite limit

$$\rho_{1,2} = 1 - e^{-2} \doteq 0.8647.$$

Using some results of Mackenzie [36], it can be proved that for $c \in \{1, 2, \cdots\}$

$$\rho_{1,c,q} \to \rho_{1,c} \quad \text{as} \quad q \to \infty,$$

where the sequence of limits $\{\rho_{1,c}\}_{c=1}^{\infty}$ remains finite as c tends to infinity. In general, $\rho_{1,c}$ must be obtained by numerically integrating an expression which arises from a recurrence technique. However, it is trivial that $\rho_{1,1} = 1$, and for $c = 2$ the required integral may be

evaluated in closed form to yield $\rho_{1,2} = 1 - e^{-2}$ in agreement with Page.

Mackenzie's work also suggests that

$$\rho_{1,c} \sim 0.7476 + \frac{0.2162}{c} + \frac{0.0360}{c^2} \quad \text{as} \quad c \to \infty. \tag{2.1}$$

The first two constants are obtained in [36] via numerical integration; however, the coefficient of c^{-2} has been estimated empirically by Mackenzie. For $c \geq 2$, (2.1) gives $\rho_{1,c}$ correct to four significant figures.

It should be noted that 0.7476 is the asymptotic packing density for arcs placed uniformly on the circle. This value, denoted by $\rho_{1,\infty}$, was first obtained by Rényi [47] and, to be more precise, it may be defined by

$$\rho_{1,\infty} = \lim_{a \downarrow 0} \rho_{\infty}(a),$$

where $\rho_{\infty}(a)$ is just the average proportion of the unit circle which is covered by a saturated packing of arcs of length a whose centers are chosen according to a uniform $(0,1]$ distribution. Blaisdell and Solomon [4] found that

$$\rho_{1,\infty} \doteq 0.74759\ 79202\ 53398$$

by developing explicit bounds which can determine fifteen significant digits correctly.

Now consider cases for which n is equal to 2, and where two points collide whenever the Lee distance between them is less than $d \in \{2, 3, \cdots\}$. This is a discrete two-dimensional analog to the one-dimensional packing problem considered previously in this section. In general, parking problems in two-dimensions are notoriously difficult and resist all attempts at mathematical solutions.

Some simulation results are summarized in Appendix B. Letting $M_{2,d,q}$ denote the number of points in a saturated packing, these results indicate that the center densities

$$r_{2,d,q} = q^{-2} E[M_{2,d,q}]$$

are approximately constant for each fixed d and various choices of $q \geq 10d$. Hence it seems

likely that

$$r_{2,d,q} \to r_{2,d} \quad \text{as} \quad q \to \infty \quad (d \geq 2)$$

for some undetermined sequence of values $r_{2,2}, r_{2,3}, r_{2,4}, \cdots$. For example, consider the eight entries having $n = 2$ and $d = 3$. These results certainly suggest that the $r_{2,3,q}$ may tend to a limit around 0.1398 as q becomes large.

Let $\hat{r}_{2,d,q}$ denote the observed average of $q^{-2}M_{2,d,q}$ for all simulation trials performed with parameters d and q, and let $\hat{r}_{2,d}$ be the average of all of the $\hat{r}_{2,d,q}$ for which $q \geq 10d$. The values $\hat{r}_{2,d}$ $(d \geq 2)$ shown below will serve as estimates of the limiting values $r_{2,d}$ $(d \geq 2)$.

Estimated limits for the center densities

d	$\hat{r}_{2,d}$
2	0.3642
3	0.1398
4	0.08025
5	0.04903
6	0.03415
7	0.02453
8	0.01882
9	0.01467
10	0.01186
11	0.009722
12	0.008176

Although it is difficult to approximate the above limiting densities analytically, several relationships among the center densities can be observed. In what follows, a scheme is developed for producing planar densities from the center densities. Then it will be shown how relationships among the limiting center densities and among the limiting planar densities can be approximated by simple functions of the parameter d. Finally, it will be shown that both approximation schemes can be used to produce estimates of an overall

limiting value $\rho_{2,d}$, and that the two methods produce estimates which agree closely with each other.

Let $\theta_{2,d} = q^2 p_{2,d,q}$, where $p_{2,d,q}$ is the probability that two points collide in the (d,q) case. Then

$$\begin{aligned}\theta_{2,d} &= 1 + 4\sum_{j=0}^{d-2}(d-1-j)\\ &= 2d(d-1)+1.\end{aligned}$$

Now suppose that each packed point serves as the position point of a "diamond" shaped configuration of $v_{2,d}$ points which are fixed relative to the position point. The contents $v_{2,2}, v_{2,3}, v_{2,4}, \cdots$ are taken to be as large as possible, subject to a constraint requiring that the diamonds surrounding two disjoint points should contain no common points. For d odd, $v_{2,d} = (d^2+1)/2$. The diamond consists of a row of d points centered on the position point, and sandwiched around this row are pairs of successively smaller rows of $d-2, d-4, \cdots, 3$, and 1 points, each centered on the position point. For d even, $v_{2,d} = d^2/2$; however, it is now not possible to have the position point being at the exact center of the configuration. The diamond consists of a row of d points, sandwiched between pairs of rows having $d-2, d-4, \cdots, 4$ and 2 points. The position point is one of the two centermost points.

Even though the diamond shaped configurations described here will not overlap if they are positioned on disjoint points, the packing problem considered here is not always the same as the problem of packing such oriented diamonds on a q by q torus shaped lattice. The problems are different if d is even. This fact can be established easily for the $d=2$ case, since it is clear that the two "diamonds" $\{(r,s),(r+1,s)\}$ and $\{(r,s+1),(r+1,s+1)\}$ do not overlap even though the Lee distance between the position points (r,s) and $(r,s+1)$ is less than two. A similar argument can be used to handle any other even valued d.

Define limiting densities $\{\rho_{2,d}\}_{d=2}^{\infty}$ by

$$\rho_{2,d} = v_{2,d}\tau_{2,d} \quad (d \geq 2).$$

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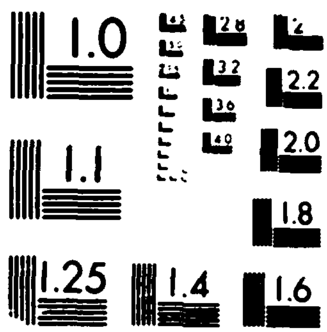

MICROCOPY RESOLUTION TEST CHART

and let

$$\hat{\rho}_{2,d} = v_{2,d}\hat{r}_{2,d} \quad (d = 2, \cdots, 12)$$

estimate $\rho_{2,d}$. The values of these estimates are given below. Note that for each even value of d, $\hat{\rho}_{2,d}$ is much closer to $\hat{\rho}_{2,d+1}$ than to $\hat{\rho}_{2,d-1}$.

Estimates of limiting planar densities

d	$\hat{\rho}_{2,d}$
2	0.72848
3	0.69876
4	0.64197
5	0.63743
6	0.61464
7	0.61331
8	0.60233
9	0.60127
10	0.59317
11	0.59305
12	0.58866

Unlike the $n = 1$ case, the $\rho_{2,d}$ are not given by an expression similar to (2.1); however, several relationships among the $\rho_{2,d}$ and the $r_{2,d}$ are apparent. Letting

$$\lambda_d = \rho_{2,d-1}/\rho_{2,d+1} \quad (d = 4, 6, 8, \cdots)$$

and

$$\phi_d = r_{2,d}/r_{2,d+1} \quad (d \geq 2),$$

it may be seen below that

$$\Lambda_d = 1 + \frac{(d+1)^2}{d^4}$$

approximates λ_d and that

$$\Phi_d = \begin{cases} \theta_{d+1}/\theta_d & (d \text{ even}) \\ 1 + \frac{2}{d} + \frac{1}{d^2} - \frac{1}{d^3} & (d \text{ odd}) \end{cases}$$

approximates ϕ_d. $\hat{\lambda}_d$ is defined by $\hat{\rho}_{2,d-1}/\hat{\rho}_{2,d+1}$, and $\hat{\phi}_d$ is defined similarly.

Comparison of approximation with estimates

d	Λ_d	$\hat{\lambda}_d$
4	1.098	1.096
6	1.038	1.039
8	1.020	1.020
10	1.012	1.014

Comparison of approximation with estimates

d	Φ_d	$\hat{\phi}_d$
2	2.600	2.606
3	1.741	1.742
4	1.640	1.637
5	1.432	1.436
6	1.393	1.392
7	1.303	1.303
8	1.283	1.284
9	1.233	1.236
10	1.221	1.220
11	1.189	1.189

The expression

$$\rho_{2,d-1}/\rho_{2,d+1} \approx 1 + d^{-4}(d+1)^2 \quad (d = 4, 6, 8, \cdots) \tag{2.2}$$

and the values $\hat{\rho}_{2,d}$ $(d = 3, 5, 7, 9, 11)$ may be used to obtain approximations for

$$\rho_{2,\infty} = \lim_{d\to\infty} \rho_{2,d}.$$

It follows from (2.2) that

$$\rho_{2,\infty} \approx \hat{\rho}_{2,2j-1} \left\{ \prod_{k=j}^{\infty} \left[1 + \frac{(2k+1)^2}{(2k)^4} \right] \right\}^{-1} \qquad (j = 2, 3, \cdots). \tag{2.3}$$

Using

$$\begin{aligned} \prod_{k=25}^{\infty} \left[1 + \frac{(2k+1)^2}{(2k^4)} \right] & \\ &> 1 + \sum_{k=25}^{\infty} \frac{4k^2 + 4k + 1}{16k^4} \\ &> 1 + \frac{1}{4} \sum_{k=25}^{\infty} \frac{1}{k^2} \\ &= 1 + \frac{1}{4} \left(\frac{\pi^2}{6} - \sum_{k=1}^{24} k^{-2} \right) \\ &> 1.010226 \end{aligned}$$

and

$$\begin{aligned} \prod_{k=25}^{\infty} \left[1 + \frac{(2k+1)^2}{(2k)^4} \right] & \\ &= \exp \left\{ \sum_{k=25}^{\infty} \log \left[1 + \frac{(2k+1)^2}{(2k)^4} \right] \right\} \\ &< \exp \left\{ \sum_{k=25}^{\infty} \left[\frac{4k^2 + 4k + 1}{16k^4} \right] \right\} \\ &< \exp \left\{ \frac{1}{4} \left(\frac{\pi^2}{6} - \sum_{k=1}^{24} k^{-2} \right) + \frac{1}{16} \int_{24}^{\infty} \left(\frac{4}{x^3} + \frac{1}{x^4} \right) dx \right\} \\ &< 1.010476 \end{aligned}$$

it is possible to obtain upper and lower bounds from (2.3) which are not too burdensome to compute. Given below are upper and lower bounds of $\rho_{2,\infty}$ based on the estimates $\hat{\rho}_{2,d}$, for d equal to 3, 5, 7, 9, and 11.

Bounds of $\rho_{2,\infty}$ obtained from (2.3)

$(d = 2j - 1)$

d	lower bound	upper bound
3	0.5656	0.5658
5	0.5664	0.5665
7	0.5655	0.5657
9	0.5654	0.5655
11	0.5644	0.5646

Alternatively,

$$\rho_{2,d}/\rho_{2,\infty} = \prod_{k=d}^{\infty}\left(\frac{v_{2,k}}{v_{2,k+1}}\Phi_k\right)$$

suggests that $\rho_{2,\infty}$ may be approximated by

$$\hat{r}_{2,d}v_{2,d}\left[\prod_{k=d}^{260000}\left(\frac{v_{2,k}}{v_{2,k+1}}\Phi_k\right)\right]^{-1}. \tag{2.4}$$

Note that the product is truncated so that it can be evaluated by computer. The estimates of $\rho_{2,\infty}$ obtained from (2.4) and the $\hat{r}_{2,d}$ $(d = 2, \cdots, 12)$ are given below.

Approximations of $\rho_{2,\infty}$ obtained from (2.4)

d	(2.4)
2	0.5671
3	0.5657
4	0.5654
5	0.5666
6	0.5650
7	0.5657
8	0.5656
9	0.5655
10	0.5641
11	0.5645
12	0.5646

Note that for each value of d, the estimate obtained from (2.4) is consonant with the upper and lower bounds provided by the previous method. Assuming that the variation in the estimates is due to chance, it seems reasonable to suppose that $\rho_{2,\infty}$ is some value close to 0.565 or 0.566. It should be noted that these estimates were obtained by assuming that the densities really do follow the pattern observed above. It is not feasible to check out the accuracy of the approximations by performing Monte Carlo experiments using extremely large values for d.

Now instead of keeping n fixed at 2, consider randomly packed 4-ary $(n, 2)$-codes for $n = 3, 4, \cdots$. Similar to the corresponding Hamming distance cases, it appears that the ratios $-\log r_{2,n}/\log\theta_{2,n}$ $(n = 3, \cdots, 7)$ are approximately constant, suggesting that the densities might be asymptotically equivalent to

$$\theta_{2,n}^{-\beta_2} \quad \text{as} \quad n \to \infty$$

for some β_2, where $\theta_{2,n} = q^n p = 2n + 1$.

Estimates of β_2

n	$-\log \hat{r}_{2,n} / \log \theta_{2,n}$
3	0.596
4	0.597
5	0.602
6	0.604
7	0.607

Likewise, the corresponding ratios for the $d = 3$ cases tend to approach a constant β_3 as n becomes large. Here $\theta_{3,n} = q^n p = 2n^2 + n + 1$.

Estimates of β_3

n	$-\log \hat{r}_{3,n} / \log \theta_{3,n}$
4	0.760
5	0.753
6	0.749
7	0.747

2.5. Other metrics

Square Box Metric

Consider packing on the space of q-ary codewords of length 2 using the metric given by

$$\mu(\mathbf{x}, \mathbf{y}) = \max\{\nu(x_1, y_1), \nu(x_2, y_2)\},$$

where ν is a metric defined in the previous section. If two points collide whenever the distance between them is less than c, then the packing sequence corresponds to packing c by c blocks of points on a q by q torus shaped lattice of points so that the packed blocks are pairwise disjoint.

For various combinations of c and q, the packing sequence described above was repeatedly simulated by computer. The results of the computer trials are summarized in Appendix C. For example, the first five cases given in Appendix C show the results of

packing 2 by 2 blocks of points on torus shaped lattices of five different sizes. The sizes range from 23 by 23 to 35 by 35. In each of the five cases, it can be seen that the average planar density observed does not differ much from the overall average of 0.747 obtained from combining the results of all of the cases.

Letting $M_{c,q}$ denote the number of points in a saturated packing, the random packing density may be defined by

$$\rho_{c,q} = (c^2/q^2)E[M_{c,q}].$$

Based on the simulation output and the knowledge that an analogous metric in a similar one-dimensional setting produces limiting densities, it seems reasonable to expect that

$$\rho_{c,q} \to \rho_c \quad \text{as} \quad q \to \infty$$

for some sequence $\{\rho_c\}_{c=2}^{\infty}$. Letting $\hat{\rho}_{c,q}$ denote the observed average of $(c^2/q^2)M_{c,q}$, ρ_c is estimated by averaging all of the $\hat{\rho}_{c,q}$ for which $q > 10c$.

Similar to the $n = 1$ case of the previous section, it is found by curve fitting that the $\hat{\rho}_c$ can be closely approximated by an expression of the form

$$\beta_0 + \beta_1 c^{-1} + \beta_2 c^{-2}. \tag{2.5}$$

Performing a least squares fit on the values $\hat{\rho}_2, \hat{\rho}_3, \cdots, \hat{\rho}_9$ yields the approximation

$$\rho_c \approx \bar{\rho}_c = 0.5626 + \frac{0.3142}{c} + \frac{0.1092}{c^2}.$$

It may be seen below that this approximation formula does rather well. Also shown below are the least square estimates based on the model

$$\rho_c = \rho_\infty \gamma^{1/c}, \tag{2.6}$$

which are given by

$$\rho_c^* = 0.5622(1.766)^{1/c}.$$

It should be noted that while two models yield fits of almost identical quality, the model given by (2.6) has one less parameter than the model given by (2.5).

Results for square box metric

c	$\hat{\rho}_c$	$\tilde{\rho}_c$	ρ_c^*	s.d. $(\hat{\rho}_c)$
2	0.7470	0.7470	0.7470	0.0007
3	0.6794	0.6795	0.6795	0.0007
4	0.6483	0.6480	0.6480	0.0007
5	0.6301	0.6298	0.6299	0.0005
6	0.6177	0.6180	0.6180	0.0005
7	0.6092	0.6097	0.6097	0.0006
8	0.6029	0.6036	0.6036	0.0007
9	0.5999	0.5989	0.5988	0.0007

Note that both models yield nearly the same value for $\rho_{2,\infty} = \lim_{c\to\infty} \rho_c$. The first model gives $\rho_{2,\infty} = 0.5626$, while the second model gives $\rho_{2,\infty} = 0.5622$. $\rho_{2,\infty}$ may be interpreted as the limiting proportion of a unit torus covered by a saturated packing of squares of area v as v tends to zero, when the centers of the squares in the packing sequence arise from a uniform distribution over the torus.

The values of the quantity

$$\Delta_2 = \sqrt{\rho_{2,\infty}} - \rho_{1,\infty}$$

produced by the two estimates of $\rho_{2,\infty}$ are 0.0025 and 0.0022. These agree closely with the estimates of Δ_2 obtained by other authors. Akeda and Hori [2], Blaisdell and Solomon [4], and Jodrey and Tory [30] estimate Δ_2 to be 0.0027, 0.0025, and 0.0021, respectively. All of these estimates serve to refute Palasti's conjecture (see [44]) that $\rho_{2,\infty} = \rho_{1,\infty}^2$. It is interesting to note that although the quantity $\rho_{2,\infty}$ discussed in the previous section may be interpreted in the same way as $\rho_{2,\infty}$ from this section, its value was estimated to be slightly larger by the methods of Section 2.4, being around 0.565 or 0.566.

Simple Cubic Metric

Now consider the simple cubic packing sequence on the space of ternary codewords

of length n. This packing scheme is described in Section 1.8, and has been discussed previously by Itoh and Udea [29] and Itoh and Solomon [27].

Letting M_n denote the number of n-tuples in a saturated packing the random packing density is given by

$$\rho_n = 2^{-n} E[M_n].$$

This follows from the fact that the content v associated with each packed point is equal to $2^n/4^n = 2^{-n}$. It can be seen below that an expression of the form

$$\alpha^n n^{-\beta} / \log n$$

fits the observed $\hat{\rho}_n$ rather well for n not too small. Performing a weighted nonlinear regression on $\hat{\rho}_7, \cdots, \hat{\rho}_{11}$ yields the approximation

$$\rho_n \approx \tilde{\rho}_n = \frac{(1.031995)^n}{n^{0.215237} \log n}.$$

The data is taken from [27]. Note that this model does not have the ratio ρ^{n+1}/ρ_n tending to 1 as suggested in [27].

Comparison for the simple cubic packing scheme

n	$\tilde{\rho}_n$	$\hat{\rho}_n$	s.d. $(\hat{\rho}_n)$
5	0.5144	0.4927	0.0010
6	0.4585	0.4508	0.0008
7	0.4214	0.4212	0.0005
8	0.3955	0.3958	0.0007
9	0.3766	0.3762	0.0012
10	0.3625	0.3631	0.0018
11	0.3519	0.3516	0.0013

2.6. Complementary codes

In Section 2, random binary (n, d)-codes were formed by adding randomly chosen words to a packed code if the selected word was at a Hamming distance of d or more from

each word already in the packed set. This section will also consider stochastically formed binary (n, d)-codes; however, now the criteria for packing new words will be different. Before describing the packing scheme and presenting the results, a few additional comments concerning error-correcting codes will be given.

For fixed n and d it is clear that information can be transmitted at the greatest rate whenever the number of words in the code is as large as possible. $A(n, d)$ denotes the number of codewords in the largest possible binary code of length n and minimum distance d. $A(n, d)$ is known precisely in some cases, while in other cases only upper and lower bounds are known. MacWilliams and Sloane [38] give relations involving $A(n, d)$, discuss the bounds on $A(n, d)$, and give values where known. Their book also serves as an excellent reference on the coding problem in general, as do the books by Peterson and Weldon [45] and Lin and Costello [35].

Several of the most familiar error-correcting codes exhibit quite a lot of structure and contain the maximum number of codewords, $A(n, d)$. Perhaps the best known of such codes is the Golay codes. This code has length 24 and minimum distance 8. It has $A(24, 8) = 4096$ codewords and the Hamming distance between any pair of codewords is either 8, 12, 16, or 24. The Golay code belongs to the class of linear codes and has numerous interesting structural properties, see Thompson [58]. Among these properties is the fact that if $\mathbf{u}$ is a codeword then so is $\mathbf{u}^*$, where $\mathbf{u}^*$ is the complement of $\mathbf{u}$, i.e. $\mathbf{u}^*$ has a one in each position where $\mathbf{u}$ has a zero and vice versa. Codes possessing this property will be called complementary codes.

For various combinations of n and d, complementary codes were stochastically generated using a computer. Each of the resulting codes possesses the additional property that the Hamming distances between all pairs of words belong to a restricted set of values. A code word of this type having words of length n can arise from the constrained random packing of binary $(n - 1)$-tuples using a procedure which will be outlined below.

Itoh (see [26], [28]) shows that it is not too difficult to stochastically generate a $(24, 8)$-code of size $A(24, 8)$, and in about 18% of the possible complementary coding schemes

with $4 \leq n \leq 12$ at least one random packing resulted in a code of size $A(n,d)$ being formed. Naturally, the unrestricted packing scheme of Section 2 may also produce codes of size $A(n,d)$; however, this was observed in only a few of the low dimensional cases. In some cases (for example, $(n,d) = (24,8)$) it has been shown that any random code of size $A(n,d)$ must be equivalent to an algebraic code of size $A(n,d)$ possessing nice properties. However, it has not been established that all random codes of size $A(n,d)$ are equivalent to algebraic codes.

Thus it may be possible to find large codes by restricted random packing, and given the close connection between error-correcting codes and the packing of n-dimensional spheres (see [34]), denser sphere packings are also possible. It is interesting to note that extremely dense packings have also been generated stochastically via simulated annealing (see [13]).

Now the packing procedure will be described. Letting $\mathcal{K}$ denote the set of allowable interword distances, require that $\mathcal{K}$ be of the form

$$\{d, d_1, \cdots, d_m, n - d_m, \cdots, n - d_1, n - d, n\}$$

where $d < d_1 < \cdots < d_m \leq n/2$. Sequentially choosing $(n-1)$-tuples at random from the set of all binary $(n-1)$-tuples, a new selection is added to the packed collection if the Hamming distances between it and all previously packed selections are elements of $\mathcal{K}$. Continuing until it is no longer possible to add to the packed set, codewords of length n are formed from the $(n-1)$-tuples by adding a zero as the nth component. These words, plus their complements, constitute an (n,d)-complementary code having the property that the distance between any pair of words belongs to $\mathcal{K}$.

Every such complementary code having $4 \leq n \leq 12$ and $2 \leq d \leq n/2$ was repeatedly generated. In each case the average observed packing density was calculated. For n and d both even, the densest codes arose from $\mathcal{K}$ containing n and all even integers between d and $n-d$ inclusive. In these cases the complementary codes formed are denser than the corresponding (n,d)-codes constructed by the method of Section 2. For all other choices

of n and d, the densest complementary codes were generated by having $\mathcal{K} = \{d, d+1, \cdots, n-d-1, n-d, n\}$. The simulation results are summarized in Appendix D. Since the random complementary coding scheme generally produces large codes more frequently than the unrestricted random coding scheme, and since for the same choice of (n, d) the complementary codes can be produced more quickly by computer (remember, $(n-1)$-tuples are being packed in the complementary case), it seems that random complementary coding may be superior to unrestricted random coding in the search for new large codes.

Simulations were avoided for the cases covered by the two results below. These facts follow from the following lemma whose simple proof is omitted.

Lemma 2.1. The Hamming distance between a codeword of even Hamming weight and a codeword of odd Hamming weight is odd. Otherwise the Hamming distance between two codewords is even.

Fact 2.1. Stochastically constructed (n, d)-complementary codes with $\mathcal{K} = \{d, d_1, \cdots, d_m, n-d_m, \cdots, n-d_1, n-d, n\}$, where $d, d_1, \cdots, d_m, n-d_m, \cdots, n-d_1$ and $n-d$ are all odd, will contain exactly four codewords.

Fact 2.2. If n is even, then a stochastically constructed $(n, 2)$-complementary code with $\mathcal{K}$ containing all even integers between 2 and $n-2$ inclusive will contain exactly $A(n, d) = 2^{n-1}$ codewords.

For the class of $(n, 2)$-codes having $\mathcal{K} = \{2, 3, \cdots, n-3, n-2, n\}$, a model similar to those of Sections 2 and 3 seems to fit the observed densities for n not too small. That is, letting $\theta_{2,n}$ equal

$$2^{n-1}\, P\{\text{two arbitrary } (n-1)\text{-tuples collide}\} = n+1,$$

it is found that the densities are reasonably approximated by

$$\theta_{2,n}^{-0.6147} - 0.1369\theta_{2,n}^{-2(0.6147)} \tag{2.7}$$

for $n \geq 9$.

Results for cases with $\mathcal{K} - \{n\} = \{2, 3, \cdots, n-2\}$

n	$\hat{r}$	(2.7)
7	0.280	0.289
8	0.272	0.268
9	0.251	0.251
10	0.235	0.236
11	0.225	0.224
12	0.212	0.212
13	0.204	0.203
14	0.193	0.194

Similarly, letting $\mathcal{K} = \{3, 4, \cdots, n-4, n-3, n\}$ produces densities that are approximated by

$$\theta_{3,n}^{-0.7753} + 4.340\theta_{3,n}^{-2(0.7753)} \tag{2.8}$$

for $n \geq 11$, where

$$\theta_{3,n} = \frac{n^2 + n + 2}{2}.$$

Results for cases with $K - \{n\} = \{3, 4, \cdots, n-3\}$

n	$\hat{r}$	(2.8)
10	0.0526	0.0462
11	0.0448	0.0450
12	0.0387	0.0385
13	0.0339	0.0335
14	0.0300	0.0305

2.7. Summary

For randomly packed q-ary (n, d)-codes, evidence has been presented which suggests that as the dimension n tends to infinity, the center densities are asymptotically equivalent to expressions of the form

$$(p/v)^{-\beta_d}.$$

Here p is the probability that two arbitrary codewords collide, and $v = q^{-n}$ is the content associated with each point in the space. This was demonstrated most convincingly for the case of binary codewords packed by Hamming distance, where additional relations involving the β_d were also found. Support was also given for some q-ary cases, and for a case where Lee distance was used as the metric instead of Hamming distance.

Relations among the packing densities were also found in cases where n was set equal to 2 and q was taken to be large. One such two-dimensional scheme yielded an estimate of the planar packing density, $\rho_{2,\infty}$, which agrees closely with estimates obtained by other investigators. The estimate is the limiting value of a formula which was shown to closely approximate the packing densities arising from a class of discrete two-dimensional packing schemes. This method of finding $\rho_{2,\infty}$ differs from those used previously. It is interesting to note that the formula is of the same form as Mackenzie's formula for the class of one-dimensional analogs. Hopefully, similar simulations may be done for the three-dimensional case, and the limiting density obtained may be compared with the results reported in Blaisdell and Solomon [5].

Chapter 3

Packing Times and Covering Times

3.1. Introduction

In Chapter 2 the time to saturation, T, was defined for a packing sequence on a metric space $\langle S, \mu \rangle$ with collision criterion δ. For a given packing sequence, T is just the total number of random independent selections from S, including rejections, required to achieve a state of saturation. For convenience, T will sometimes be called the packing time.

It is also possible to define another random variable associted with the process of sequentially selecting points from the space S of a metric space $\langle S, \mu \rangle$. For each element $s \in S$, define a coverage set $A(s)$ such that if X is uniformly distributed over S then $P\{X \in A(s)\} = \epsilon$, for some $\epsilon > 0$. $A(s)$ can be a neighborhood of s, but this is not necessary. Suppose that $C_1, C_2, \cdots$ are chosen uniformly and independently from S and let W, called the covering time, be defined by

$$W = \min \left\{ k \in \{1, 2, \cdots\} : S \subset \bigcup_{i=1}^{k} A(C_i) \right\}.$$

It is said that S is covered by $\bigcup_{i=1}^{j} A(C_i)$ for any $j \geq W$. Equivalently, it is also said that the sequence of points $C_1, C_2, \cdots, C_j$ $(j \geq W)$ provide a covering of the space. Random coverage problems are abundant in the literature, see for example [7], [16], [17], [18], [19], [20], [24], [25], [40], [42], [49], [50], [51], [54], [57], and [59].

Below, packing and covering sequences are examined on several spaces. In each space, coverage sets are defined so that for some choice of μ and δ the packing sequence corresponds to randomly packing the coverage sets so that members of the packed collection

are pairwise disjoint. The distributions of the random variables T and W are compared in each case.

3.2. The Continuous Circle

Consider the sequential random packing of arcs of length $a < 1$ on a circle of unit circumference. Random points $C_1, C_2, \cdots$ sequentially chosen from $S = (0, 1]$ serve as the midpoints of arcs of length a. The collision criterion, δ, is set equal to a, and μ is taken to be the metric described in Section 1.4. For this setting, Rényi [47] has shown that the expected proportion of the circle covered by packed arcs at time T approaches the constant

$$\int_0^\infty \exp\left\{-2\int_0^t \frac{1-e^{-u}}{u} du\right\} dt$$

as the length of the arcs tends to zero.

It will now be shown that $E[T] = \infty$ whenever $a \leq \frac{1}{4}$. To do this it is convenient to first define a few terms. When a second arc is fairly packed on the circle, two gaps having lengths summing to $1 - 2a$ are formed. Let G denote the shorter of the two gaps, and let L denote its length. Note that L is uniformly distributed over $\left(0, \frac{1}{2} - a\right)$.

Now define a random variable U as follows. If $L \leq a$, let $U = 0$. If $L > a$, let $U = k$ if C_k is the midpoint of the first arc which is packed into G.

Note that the first random arc will always be packed onto the circle, and that the expected number of additional selections needed to pack a second arc onto the circle is $\frac{1}{1-2a}$. Then given that $L = x > a$, the probability that a random arc is packed into G is just $x - a$. Hence, for $x > a$ it follows that

$$\begin{aligned} E[U \mid L = x] &= 1 + \frac{1}{1-2a} + \frac{1}{x-a} \\ &> \frac{1}{x-a}. \end{aligned}$$

Clearly $T \geq U$, so that for any $a < \frac{1}{4}$ and any ε between 0 and $\frac{1}{2} - 2a$ it follows that

$$\begin{aligned} E[T] &\geq E[U] \\ &= \int_0^{\frac{1}{2}-a} E[U \mid L = x] \frac{1}{\left(\frac{1}{2}-a\right)} dx \\ &> \frac{1}{\left(\frac{1}{2}-a\right)} \int_{a+\varepsilon}^{\frac{1}{2}-a} \frac{1}{x-a} dx \\ &= \frac{1}{\left(\frac{1}{2}-a\right)} \log\left(\frac{\frac{1}{2}-2a}{\varepsilon}\right). \end{aligned}$$

Since this inequality is true for all $\varepsilon \in (0, \frac{1}{2} - 2a)$ it follows that $E[T] = \infty$. Thus the following theorem has been proved for the case of $a < \frac{1}{4}$. The proof for the remaining values of a is similar.

Theorem 3.1. For the packing sequence of arcs of length a on a circle of unit circumference, if $a < \frac{1}{3}$ then $E[T] = \infty$.

Now consider the sequential covering of the circle with random arcs of length a, and let $M(a) = [\![a^{-1}]\!]$. Stevens [57] determined that

$$P\{W \leq k\} = \sum_{j=0}^{M(a)} (-1)^j \binom{k}{j} (1 - ja)^{k-1}. \tag{3.1}$$

and Flatto and Konheim [17] used (3.1) to show that

$$E[W] = 1 - \sum_{k=1}^{M(a)} (-1)^k \frac{(1-ka)^{k-1}}{(ka)^{k+1}}$$

and

$$E[W] \sim a^{-1} \log a^{-1} \quad \text{as} \quad a \downarrow 0. \tag{3.2}$$

Adopting the convention that the ratio of any finite number over infinity is equal to zero, note that for any arclength $a < \frac{1}{3}$,

$$\frac{E[W]}{E[T]} = 0.$$

Now consider randomly packing and covering the circle with arcs of variable length. That is, suppose arc midpoints $C_1, C_2, \cdots$ are selected from $S = (0, 1]$ as before, and let

the arc centered on C_i have random length Z_i. The lengths $Z_1, Z_2, \cdots$ are taken to be i.i.d. random variables having c.d.f. F.

The coverage problem can be described as before, only now the coverage sets

$$A_i = \{s \in S : \mu(s, C_i) < Z_i/2\} \qquad (i = 1, 2, \cdots)$$

have random size. The packing problem can be described as follows. The coverage set A_i is added to the collection of packed sets if and only if it does not intersect any previously packed member of the collection. The packed collection is considered to be saturated whenever the probability that an additional member can be added is zero.

Siegel and Holst [51] derived an expression for the probability that the circle is completely covered by arcs of variable length. For the case of

$$F(x) = x \qquad (0 < x < 1),$$

their result may be written

$$P\{W \leq k\} = 1 + \frac{k!}{(2k-1)!} \sum_{j=1}^{k} (-1)^j b_j \tag{3.3}$$

where

$$b_j = \frac{1}{(k-j)! j 2^{k-j}} \sum \binom{k-j}{m_1, \cdots m_j} \prod_{i=1}^{j} (2m_i + 1)!$$

with the sum being over all sets of non-negative integers $m_1, \cdots, m_j$ such that $\sum_{i=1}^{j} m_i = k - j$. For this case of random arcs having mean length $\frac{1}{2}$, it may be shown that $P\{W \leq k\}$ is greater than the corresponding probability for arcs of fixed length $\frac{1}{2}$. This is accomplished by noting that the first two terms of (3.3) equals $P\{W \leq k\}$ for the arcs of equal length, and then showing that the sequence $b_2, b_3, \cdots$ constitutes a monotonically decreasing sequence. It follows that $E[W] < \infty$ for the random arc lengths, since the expectation is finite for the constant length arcs.

For $F(x) = x$ it is clear that with probability 1 the packed set will not become saturated. This follows from the fact that with probability 1 each newly packed arc will

be situated such that there are open gaps of positive length on both sides of it, and with positive probability an additional arc of sufficiently small size can be packed into each gap.

Now suppose that F is such that

$$P\{Z_i \geq a\} = 1$$

for some $a > 0$. Then it is clear that $E[W] < \infty$ since the expectation is finite in the case of arcs having constant length a, because the arcs of variable length will cover the circle at least as quickly as the arcs of length a do.

It is still possible to have $E[T]$ being infinite in this case. For instance, suppose F is the uniform distribution on (a,b), where $0 < a < b \leq \frac{1}{4}$. Then with positive probability the second arc selected will form a gap G of length $L \in (a,b)$, and given that this event occurs, L will be uniformly distributed over (a,b). Given that $L = x \in (a,b)$, the probability that an arbitrary arc will be packed into G is

$$\frac{1}{(b-a)} \int_a^x (x-z)dz = \frac{(x-a)^2}{2(b-a)}.$$

It then follows that the expected number of attempts required to pack the gap with an arc, given that $a < L < b$, is

$$\frac{1}{(b-a)} \int_a^b \frac{2(b-a)}{(x-a)^2} dx = \infty.$$

These facts are sufficient to ensure that $E[T] = \infty$.

3.3. Interarrival times

Let the random variables $T_1, T_2, \ldots$ be defined as follows. If saturation occurs before the kth point is added to the packed set, then $T_k = \infty$. Otherwise, $T_k = j$ if C_j is the kth point packed.

Now define interarrival times $U_1, U_2, \cdots$ by

$$U_1 = T_1$$

and

$$U_k = \begin{cases} T_k - T_{k-1} & \text{if } T_k < \infty \\ \infty & \text{if } T_k = \infty. \end{cases}$$

Note that $P\{U_1 = 1\} = 1$ and

$$P\{U_2 = k\} = p^{k-1}(1-p) \qquad (k = 1, 2, \cdots),$$

where

$$p = P\{C_1 \wedge C_j\} \qquad (j \neq 1).$$

For the case of packing arcs of length a on a circle of unit circumference,

$$E[U_1] = 1,$$

$$E[U_2] = \frac{1}{1-2a} \quad (a < 1/2),$$

and for $a < 1/4$

$$\begin{aligned} E[U_3] &= \left(\frac{1}{2} - a\right)^{-1} \left(\int_0^a \frac{dx}{1-3a-x} + \int_a^{\frac{1}{2}-a} \frac{dx}{1-4a} \right) \\ &= \frac{2\log\left(\frac{1-3a}{1-4a}\right) + 1}{(1-2a)} \\ &= \frac{1}{1-4a+a^2+O(a^3)}. \end{aligned}$$

For $a < 1/6$

$$E[U_4] = \left(\frac{1}{2} - a\right)^{-1} \left(\int_0^a A(x)dx + \int_a^{2a} B(x)dx + \int_{2a}^{3a} \frac{C(x)dx}{1-4a} + \int_{3a}^{\frac{1}{2}-a} D(x)dx \right).$$

where

$$A(x) = (1-3a-x)^{-1} \left(1 + 2\int_0^a \frac{dy}{1-4a-x-y} \right),$$

$$B(x) = (1-4a)^{-1} \left(\frac{1-5a-x}{1-6a} + \frac{x-a}{1-3a-x} + 2\int_0^a \frac{dy}{1-5a-y} \right),$$

$$C(x) = \frac{1-5a-x}{1-6a} + 2\left(\int_{(\frac{x-a}{2})}^a \frac{dy}{1-3a-x} + \int_a^{x-a} \frac{dy}{1-4a-x+y} \right) + 2\int_0^a \frac{dy}{1-5a-y}.$$

and

$$D(x) = (1-4a)^{-1} \left(\frac{1-8a}{1-6a} + 4\int_0^y \frac{dy}{1-5a-y} \right).$$

It may be shown that

$$E[U_4] = 1 + 6a + 33a^2 + O(a^3),$$

and also it is true that

$$E[U_4] = \frac{1}{1 - 6a + 3a^2 + O(a^3)}.$$

For $k \geq 5$, an expression for $E[U_k]$ is difficult to obtain. For fixed k,

$$E[U_k] = \frac{1}{1 - 2(k-1)a + O(a^2)};$$

however, for any particular value $a < [2(k-1)]^{-1}$ the approximation

$$\frac{1}{1 - 2(k-1)a}$$

overestimates $E[U_k]$.

3.4. The discrete circle

Consider the sequential random packing of arcs of length c on the discrete circle of circumference n. Points $C_1, C_2, \cdots$ are uniformly and independently selected from $S = \{1, 2, \cdots, n\}$. Letting

$$\mu(x, y) = \begin{cases} |x - y| & \text{if } |x - y| \leq n/2 \\ n - |x - y| & \text{otherwise,} \end{cases}$$

two arcs with clockwise endpoints C_i and C_j will overlap if and only if

$$\mu(C_i, C_j) < c.$$

Similarly, the circle can be sequentially covered with random arcs of length c by letting the coverage sets be given by

$$A(s) = \{s, s \ominus 1, \cdots, s \ominus (c-1)\},$$

where the subtraction is modulo n.

Note that for $c = 1$ the circle is packed if and only if the circle is covered. Thus the random variables T and W have the same distribution. Since the circle is not covered until each member of S has been selected at least one time, and because the waiting times

between the initial occurrences of elements of S have geometric distributions, an easy argument yields that

$$E[W] = \sum_{k=0}^{n-1} \frac{n}{n-k}.$$

It follows that

$$E[W] \sim n \log n \quad \text{as} \quad n \to \infty. \tag{3.4}$$

Since $E[W]$ is finite,

$$\frac{E[W]}{E[T]} = 1$$

for the case of $c = 1$.

The distribution of W, or equivalently of T, may be determined in this case by making use of the method of inclusion-exclusion. It is found that

$$P\{W \le k\} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (1 - j/n)^k \quad (k \ge n),$$

and further, if n is not too small, it can be seen that the approximating expression

$$P\{W \le k\} \approx \exp(-ne^{-\frac{k}{n}})$$

does rather well (see [14]). The related case with selections from S not being uniform is discussed by Flatto and Newman [18]; however, they do not obtain the distribution of W exactly.

For $c \ge 2$ the distributions of T and W are difficult to obtain. The determination of $P\{W > k\}$ by a scheme analogous to Stevens' method for the continuous circle becomes unwieldy due to the fact that now two or more arcs may exactly coincide with positive probability. However, upper and lower bounds for $P\{W \le k\}$ and $E[W]$ may be obtained via a method based upon an idea of Cooke [7].

Consider a circle of circumference $n = cm$ for some integer m, and let $G_1, G_2, \cdots, G_m$ be a partition of S given by

$$G_j = \{(c-1)j + 1, (c-1)j + 2, \cdots, cj - 1, cj\} \quad (j = 1, \cdots, m).$$

Note that if S is covered by k random arcs of length c, then each set in the partition must contain at least one clockwise endpoint C_i ($i \in \{1, \cdots, k\}$). So if W_G denotes the first time for which each member of the partition contains at least one endpoint, it follows that $W_G \leq W$. This fact and the principle of inclusion-exclusion yield that

$$\begin{aligned} P\{W \leq k\} &\leq P\{W_G \leq k\} \\ &= \sum_{j=0}^{m} (-1)^j \binom{m}{j} (1 - j/m)^k \quad (k \geq m). \end{aligned}$$

Note that W_G has the same distribution as W did for the case where $c = 1$ and $S = \{1, \cdots, m\}$. Hence it follows that

$$\begin{aligned} E[W] &\geq E[W_G] \\ &= \sum_{k=0}^{m-1} \frac{m}{m-k}, \end{aligned}$$

which is asymptotically equivalent to $m \log m$ as m tends to infinity. Since $m = n/c$ it follows that $E[W]$ has a lower bound which is asymptotically equivalent to

$$\frac{n}{c} \log \frac{n}{c} \quad \text{as} \quad n \to \infty.$$

Now consider a circle of circumferene $n = \left(\left[\left[\frac{c+1}{2}\right]\right]\right) m$ ($m \in \{1, 2, \cdots\}$), and a partition of S given by

$$H_j = \{(c' - 1)j + 1, (c' - 1)j + 2, \cdots, c'j - 1, c'j\} \quad (j = 1, \cdots, m)$$

where $c' = \left[\left[\frac{c+1}{2}\right]\right]$. Note that if each set in the partition contains at least one clockwise endpoint, then the circle is completely covered by arcs of length c. So if W_H denotes the first time for which each member of the partition contains at least one endpoint, then $W_H \geq W$. Thus

$$\begin{aligned} P\{W \leq k\} &\geq P\{W_H \leq k\} \\ &= \sum_{j=0}^{m} (-1)^j \binom{m}{j} (1 - j/m)^k \quad (k \geq m), \end{aligned}$$

and

$$E[W] \le E[W_H]$$
$$= \sum_{k=0}^{m-1} \frac{m}{m-k}.$$

This implies that $E[W]$ has an upper bound which is asymptotically equivalent to

$$\frac{n}{[[\frac{c+1}{2}]]} \log \frac{n}{[[\frac{c+1}{2}]]} \quad \text{as} \quad n \to \infty.$$

Consider again the circle of circumference $n = cm$, the partition $G_1, G_2, \cdots, G_m$, and the random variable W_G defined previously. Note that for a packing sequence of random arcs of length c, saturation cannot occur until at least one point has been selected from every set in the partition. Hence $W_G \le T$ and $E[T] \ge E[W_G]$. Thus $E[T]$ has a lower bound which is asymptotically equivalent to

$$\frac{n}{c} \log \frac{n}{c} \quad \text{as} \quad n \to \infty.$$

An upper bound for $E[T]$ can be obtained as follows. Note that whenever an element $s \in S = \{1, \cdots, n\}$ is selected as a clockwise endpoint of an arc, either the arc will be rejected or the arc will be added to the packed collection of arcs. If the arc is rejected, then subsequent selections of that element s will also lead to rejection. If the arc is packed, then subsequent selections of the same element will lead to rejection. Hence it is clear that as soon as each element of S has been selected at least once then no further arcs will be added to the packed collection. Letting W_n denote the first time for which each element of S has been selected at least once, it then follows that $T \le W_n$. Thus $E[T] \le E[W_n]$, and so $E[T]$ has an upper bound which is asymptotically equivalent to

$$n \log n \quad \text{as} \quad n \to \infty.$$

Note that the upper and lower bounds for $E[W]$ and $E[T]$ do not indicate which expectation is smaller. Since it is the case that $W < T$ for some outcomes of $C_1, C_2, \cdots$ while $W > T$ for others, a simulation study was done to see whether or not a relation like $E[W] \le E[T]$ seems to hold in general. 2000 trials were performed for each of the covering

cases, and 1000 trials were done for each of the packing cases. The results are shown in the two tables below.

c	n	$\bar{W}$	$\bar{T}$
2	200	580.4425	785.470
2	300	938.2030	1307.250
2	400	1311.2655	1829.266
2	600	2079.9205	2992.124
2	800	2894.2290	–

c	n	$\bar{W}$	$\bar{T}$
3	300	611.3685	1006.389
3	450	979.7920	1694.828
3	600	1371.6885	2417.914
3	900	2174.7680	4018.000
3	1200	3032.0445	–

In every case above, it appears that $E[W] < E[T]$; however, the case of $n = 8$ and $c = 2$ shows that this is not true in general since there direct calculations give $E[W] = 151/15$ and $E[T] = 147/15$. It may also be seen from the results above, that for n sufficiently large, $E[W]$ appears to be proportional to

$$n \log\left(\frac{3n}{c}\right) \tag{3.5}$$

and $E[T]$ appears to be proportional to

$$n \log\left(\frac{n}{c^2}\right). \tag{3.6}$$

Evidence for these observations is presented below. A few additional covering results for $c = 4$ and $c = 5$ have been added.

c	n	$\bar{W}/(3.5)$	$\bar{T}/(3.6)$
2	200	0.509	1.004
2	300	0.512	1.009
2	400	0.512	0.993
2	600	0.510	0.995
2	800	0.510	–
3	300	0.357	0.957
3	450	0.356	0.963
3	600	0.357	0.960
3	900	0.355	0.969
3	1200	0.356	–
4	400	0.278	–
4	600	0.278	–
4	800	0.279	–
5	500	0.225	–
5	750	0.226	–
5	1000	0.225	–

The interarrival times for adding new arcs to the packed collection are similar to those for the continuous circle, and they provide useful anchor points for checking the simulation programs. For the continuous case, where $p = 2a$, it can be shown that

$$E[U_1] = 1,$$

$$E[U_2] = \frac{1}{1-p},$$

$$E[U_3] = \frac{1}{1 - 2p(1 - p/8) + O(p^3)}$$

and

$$E[U_4] = \frac{1}{1 - 3p(1 - p/4) + O(p^3)}.$$

For the discrete circle case, where $p = (2c - 1)/n$,

$$E[U_1] = 1 \qquad (n \geq c),$$
$$E[U_2] = \frac{n}{n - (2c - 1)} = \frac{1}{1 - p} \qquad (n \geq 2c),$$

and for $n > 4(c - 1)$

$$E[U_3] = \frac{n - 4c + 3}{n - (2c - 1)} \frac{n}{n - (4c - 2)} + \frac{2}{n - (2c - 1)} \sum_{j=0}^{c-2} \frac{n}{n - (3c - 1 + j)}.$$

For $c = 2, 3, 4, 5$ it can be shown that

$$E[U_3] = \frac{1}{1 - 2p(1 - p/8) + O(p^3)}$$

as was true for the continuous case. It is not unreasonable to expect that this expression holds for other values of c as well.

For $c = 2$ it can be shown that

$$\begin{aligned} E[U_4] &= \frac{n}{n - (9 - 6/n) + O(n^{-2})} \\ &= \frac{1}{1 - 3p\left(1 - \frac{2}{9}p\right) + O(p^3)}, \end{aligned}$$

and for $c = 3$ it can be seen that

$$\begin{aligned} E[U_4] &= \frac{n}{n - (15 - 18/n) + O(n^{-2})} \\ &= \frac{1}{1 - 3p\left(1 - \frac{6}{25}p\right) + O(p^3)}. \end{aligned}$$

These results suggest that

$$\frac{1}{1 - 3p(1 - p/4)}$$

might approximate $E[T_4]$ reasonably well for $c \geq 4$ and n not too small.

3.5. Multidimensional spaces

Consider the metric spaces $\langle S_n, \mu_n \rangle$ $(n = 1, 2, \cdots)$, where

$$S_n = \{(u_1, \cdots, u_n) : u_i \in \{1, 2, 3\}\}$$

and μ_n is the metric given by

$$\mu_n(\mathbf{x}, \mathbf{y}) = \max_{1 \le i \le n} |x_i - y_i|.$$

These simple cubic metric spaces are equivalent to the ones described previously in section 1.8 and section 2.5. The collision criterion, δ, is always taken to be 2.

The packing sequence corresponds to letting the sequence $C_1, C_2, \cdots$ of points from S_n represent the centers of n-dimensional cubes of sidelength 2. The sides of these cubes are aligned with the sides of the n-dimensional cube of sidelength 4 having vertices $\{(u_1, \cdots, u_n) : u_i \in \{0, 4\}\}$. Since two random points C_i and C_j collide if and only if $\mu_n(C_i, C_j) < 2$, it follows that two points are disjoint only if their surrounding boxes of sidelength 2 do not overlap.

The coverage sequence can be described as follows. Let $\mathcal{U}_n$ denote the n-dimensional cube of edgelength 4 which contains the 3^n elements of S_n in its interior. Then letting $\mathcal{A}(C_i)$ denote the box of sidelength 2 centered on C_i, the covering time W is the smallest integer w for which

$$\mathcal{U}_n = \bigcup_{i=1}^{w} \mathcal{A}(C_i).$$

Note that this definition of W differs slightly from the one given in section 3.1, because the space being covered, $\mathcal{U}_n$, is not the same as S_n.

Let S'_n denote the set containing the 2^n corner sites in $\mathcal{U}_n$. Thus

$$S'_n = \{(u_1, \cdots, u_n) : u_i \in \{1, 3\}\}.$$

Note that $\mathcal{U}_n$ is covered if and only if each member of S'_n has been selected at least one time. Hence it follows that

$$\begin{aligned} E[W] &= \sum_{k=0}^{2^n - 1} \frac{3^n}{(2^n - k)} \\ &\sim 3^n \log 2^n \quad \text{as} \quad n \to \infty. \end{aligned}$$

Now let M_n denote the number of points in a saturated packing. Note that just before the M_nth point was packed, there was at least one site available in which a point

could be packed. Similarly, just before the $(M_n - 1)$th point was packed there were at least two available sites, and in general there are $k + 1$ or more available sites just previous to the packing of the $(M_n - k)$th point.

For $0 \leq k \leq M_n - 1$, let Z_n be the number of trials required to place the $(M_n - k)$th packed point, and for $M_n \leq k < 2^n$ let Z_n equal $\frac{3^n}{(k+1)}$. It follows from comments above that

$$E[Z_k] \leq \frac{3^n}{(k+1)}.$$

Noting that $1 \leq M_n \leq 2^n$, it also follows that

$$T \leq \sum_{k=0}^{2^n-1} Z_k$$

and so

$$\begin{aligned} E[T] &\leq \sum_{k=0}^{2^n-1} E[Z_k] \\ &\leq \sum_{k=0}^{2^n-1} \frac{3^n}{(k+1)} \\ &= E[W]. \end{aligned}$$

It is also possible to show that $E[T] < E[W]$ and $T \leq W$ almost surely. A thorough proof of this last fact is rather tedious, but not difficult.

For $1 \leq n \leq 5$ $E[W]$ was calculated exactly. For $1 \leq n \leq 2$, $E[T]$ was calculated exactly, and for $3 \leq n \leq 5$ $E[T]$ was estimated using simulation results. For $3 \leq n \leq 4$, 50,000 trials were done in each case, and 10,000 trials were performed for the case $n = 5$. These results are presented below.

n	$E[W]$	$E[T]$	$E[W]/E[T]$
1	4.5	3	1.5
2	18.75	10.2	1.84
3	73.38	37.60	1.95
4	273.8	142.3	1.92
5	986.2	541.2	1.82

Cooke [7] has considered sequential coverage sequences for other multidimensional spaces. From Miles' results dealing with Poisson point processes (see [40] and [41]), Cooke arrives at expressions of the form

$$E[W] \sim \gamma_1\, p \log \gamma_2\, p \quad \text{as} \quad p \downarrow 0,$$

where p is the probability that two random points collide and γ_1 and γ_2 are constants. It should be noted that Cooke does not offer direct proofs of some of his results, he only claims that they are suggested by the work of Miles.

For the case of covering the unit two-dimensional torus with random disks of radius a, it follows form one of Cooke's claims that

$$E[W] \approx \frac{1}{\pi a^2} \log \frac{1}{a^2} \tag{3.7}$$

for a not too large. The accuracy of this approximation may be checked by comparing (3.7) with simulation results for various choices of a. In the simulation runs, coverage of the torus was checked by making sure that each crossing was covered. See [54] for details of this method in a similar setting.

a	no. trials	observed	(3.7)
0.25	20	23.00	14.12
0.20	20	47.80	25.62
0.15	20	97.15	53.68
0.10	10	251.6	146.6

The results above indicate that Cooke's formula does not provide very good approximations. Thus it is probably not appropriate to make quick substitutions in results derived for a homogeneous planar Poisson point process as Cooke did. The work of Domb [11] also indicates that it is not trivial to obtain sequential coverage results from those derived for a Poisson point process.

$E[W]$ can be determined exactly in the case of covering the surface of a sphere with random hemispheres. S is taken to be the surface of a sphere having unit radius, and

$C_1, C_2 \cdots$ are the centers of spherical caps of half angle radius $\frac{\pi}{2}$. For this setting it follows as a special case of a result of Wendel [59] that

$$P\{W > k\} = 2^{-k}(k^2 - k + 2) \qquad (k \geq 1).$$

Hence

$$\begin{aligned} E[W] &= 1 + \sum_{k=1}^{\infty} 2^{-k}(k^2 - k + 2) \\ &= 7. \end{aligned}$$

For the related lower dimensional problem of covering a circle with hemispheres, it follows from Wendel's result that

$$\begin{aligned} E[W] &= 1 + \sum_{k=1}^{\infty} \left(\frac{k}{2^{k-1}}\right) \\ &= 5. \end{aligned}$$

The corresponding packing problems are trivial for both of these cases.

Now consider covering the surface of a sphere in 3-space with spherical caps of half angle radius $\alpha < \frac{\pi}{2}$. The results of a simulation study (see [54]) indicate that $P\{W > k\}$ is closely approximated by a formula which is a special case of a result of Miles [40]. Letting $M(\alpha)$ denote the smallest number of caps for which a covering is possible, an improved approximation is

$$P\{W > k\} \approx \begin{cases} 1 & \text{for } k < M(\alpha) \\ 2^{-k}k(k-1)\sin^2\alpha(1+\cos\alpha)^{k-2} & \text{for } k \geq M(\alpha). \end{cases}$$

Hence it follows that

$$\begin{aligned} E[W] &\approx M(\alpha) + \frac{\sin^2\alpha}{4} \sum_{k=M(\alpha)}^{\infty} k(k-1)\left(\sin^2\frac{\alpha}{2}\right)^{k-2} \\ &\sim M(\alpha)\left[1 + \frac{\sin^2\alpha}{4} M(\alpha)\left(\sin^2\frac{\alpha}{2}\right)^{M(\alpha)-2}\right] \quad \text{as} \quad \alpha \downarrow 0. \end{aligned}$$

It should be noted that $M(\alpha)$ is not easy to determine in general.

The distributions of T for packing sequences on spherical surfaces and tori are difficult to determine. For the packing of n-dimensional cubes in an n-dimensional torus, a straightforward extension of the argument for the one-dimensional case yields that $E[T] = \infty$ provided that the cubes are not too large.

3.6. Summary

Although covering problems are abundant in the literature, sequential covering results concerning $E[W]$ have been considered by only a few authors, including Flatto and Cooke. Similarly, results concerning T haven't been previously developed even though numerous investigators have studied random packing densities.

Among the results in this chapter, it has been shown that the ratio $E[W]/E[T]$ can be equal to zero, less than one, equal to one, or greater than one. The ratio is equal to zero for the continuous circle since $E[T] = \infty$. For the discrete circle, the ratio equals one if $c = 1$, and is less than one if $c \geq 2$ and n is not too small. The ratio exceeds one in the simple cubic case discussed in the previous section.

REFERENCES

[1] Abramson, M. and Moser, W. O. J. (1970). More birthday surprises. *American Math. Monthly* **77**, 856-858.

[2] Akeda, Y. and Hori, M. (1976). On random sequential packing in two and three dimensions. *Biometrika* **63**, 361-366.

[3] Billingsley, P. (1979). *Probability and Measure,* John Wiley, New York.

[4] Blaisdell, B. E. and Solomon, H. (1970). On random sequential packing in the plane and a conjecture of Palásti. *J. Appl. Prob.* **7**, 667-698.

[5] Blaisdell, B. E. and Solomon, H. (1982). Random sequential packing in Euclidean spaces of dimensions three and four and a conjecture of Palásti. *J. Appl. Prob.* **19**, 382-390.

[6] Blaum, M., Eisenberger, I., Lorden, G., and McEliece, R. J. (1984). More about the birthday surprise. Technical report, California Institute of Technology, Department of Mathematics, Pasadena, California.

[7] Cooke, P. J. (1974). Bounds for coverage probabilities with applications to sequential coverage problems. *J. Appl. Prob.* **11**, 281-293.

[8] de Finetti, B. (1964). Alcune ossevazioni in tema de "suddivisione casuale." *Giornale Istituto Italiano degli Attuari* **27**, 151-173.

[9] Diaconis, P. (private communications, some pertaining to material which will be included in *Coincidences*, a book being written with F. Mosteller)

[10] Dixon, W. J. (Ed.) (1981). *BMDP Statistical Software 1981.* University of California Press, Berkeley.

[11] Domb, C. (1947). The problem of random intervals on a line. *Proc. Cambridge Philos. Soc.* **43**, 329-341.

[12] Downton, F. (1961). A note on vacancies on a line. *J. Roy. Stat. Soc. Ser. B* **23**, 207-214.

[13] El Gamal, A. A., Hemachandra, L. A., Shperling, I., and Wei, V. K. (1987). Using simulated annealing to design good codes. *IEEE Trans. Inform. Theory* **33**, 116-123.

[14] Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*, vol. 1, 3rd. ed. , John Wiley, New York.

[15] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications*, vol. 2, 2nd. ed. , John Wiley, New York.

[16] Flatto, L. (1973). A limit theorem for random coverings of a circle. *Israel J. Math.* **15**, 167-184.

[17] Flatto, L. and Konheim, A. G. (1962). The random division of an interval and the random covering of a circle. *SIAM Rev.* **4**, 211-222.

[18] Flatto, L. and Newman, D. J. (1977). Random coverings. *Acta Math.* **138**, 241-264.

[19] Gilbert, E. N. (1965). The probability of covering a sphere with N circular caps. *Biometrika* **52**, 323-330.

[20] Hall, P. (1985). On the coverage of k-dimensional space by k-dimensional spheres. *Ann. Probab.* **13**, 991-1002.

[21] Hoeffding, W. (1948). A class of statistics with asymptotically normal distibution. *Ann. Math. Statist.* **19**, 293-325.

[22] Holst, L. (1980). On the lengths of the pieces of a stick broken at random. *J. Appl. Prob.* **17**, 623-634.

[23] Holst, L. (1981). On convergence of the coverage by random arcs on a circle and the largest spacing. *Ann. Probability* **9**, 648-655.

[24] Huffer, F. (1986). Variability orderings related to coverage problems on the circle. *J. Appl. Prob.* **23**, 97-106.

[25] Hüsler, J. (1982). Random coverage of the circle and asymptotic distributions. *J. Appl. Prob.* **19**, 578-587.

[26] Itoh, Y. (1986). Golay code and random packing. *Ann. Inst. Statist. Math. Part B* **38**, 583-588.

[27] Itoh, Y. and Solomon, H. (1986). Random sequential coding by Hamming distance. *J. Appl. Prob.* **23**, 688-695.

[28] Itoh, Y. and Solomon, H. (1986). A stochastic construction of Golay-Leech packing. Unpublished manuscript.

[29] Itoh, Y. and Ueda, S. (1983). On packing density by a discrete random sequential packing of cubes in a space of n dimensions. *Proc. Inst. Statist. Math.* **31**, 65-69 (in Japanese with English summary).

[30] Jodrey, W. S. and Tory, E. M. (1980). Random sequential packing in R^n. *J. Statist. Comput. Simul.* **10**, 87-93.

[31] Klamkin, M. S. and Newman, D. J. (1967). Extensions of the birthday surprise. *Journal Combinatorial Theory* **3**, 279-282.

[32] Kozlov, M. V. (1970). On random subsets of vertices of an n-dimensional cube. *Soviet Math. Dokl.* **11**, 21-24.

[33] Laplace, P. S. (1820). *Théorie Analytique des Probabilités (3rd ed.)*, Courcier, Paris.

[34] Leech, J. and Sloane, N. J. A. (1971). Sphere packings and error-correcting codes. *Can. J. Math.* **23**, 718-745.

[35] Lin, S. and Costello, Jr., D. J. (1983). *Error Control Coding: Fundamentals and Applications*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

[36] Mackenzie, J. K. (1962). Sequential filling of a line by intervals placed at random and its application to linear adsorption. *J. Chem. Phys.* **37**, 723-728.

[37] McKinney, E. H. (1966). Generalized birthday problem. *American Math. Monthly* **73**, 385-387.

[38] McWilliams, F. J. and Sloane, N. J. A. (1977). *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam, New York, Oxford.

[39] Marsaglia, G. (1961). Uniform distributions over a simplex. Mathematical Note No. 250, Math. Research Lab. , Boeing Scientific Research Lab. Report #D1-82-0152.

[40] Miles, R. E. (1969). The asymptotic values of certain coverage probabilities. *Biometrika* **56**, 661-680.

[41] Miles, R. E. (1970). On the homogeneous planar Poisson point process. *Math. Biosci.* **6**, 85-127.

[42] Moran, P. A. P. and Fazekas de St. Groth, S. (1962). Random circles on a sphere. *Biometrika* **49**, 389-396.

[43] Page, E. S. (1959). The distribution of vacancies on a line. *J. Roy. Stat. Soc. Ser. A* **21**, 364-374.

[44] Palásti, I. (1960). On some random space filling problems. *Publ. Math. Inst. Hung. Acad. Sci.* **5**, 353-359.

[45] Peterson, W. W. and Weldon, Jr., E. J. (1972). *Error-Correcting Codes*. 2nd. ed., The MIT Press, Cambridge, Massachusetts.

[46] Pyke, R. (1965). Spacings (with discussion). *J. R. Stat. Soc.* **B 27**, 395-449.

[47] Rényi, A. (1958). On a one dimensional problem concerning space filling. *Publ. Math. Inst. Hung. Acad. Sci.* **3**, 109-127.

[48] Sevast'yanov, B. A. (1972). Poisson limit law for a scheme of sums of dependent random variables. *Theory of Probability and its Applications* **17**,695-699.

[49] Siegel, A. F. (1978). Random space filling and moments of coverage in geometrical probability. *J. Appl. Prob.* **15**, 340-355.

[50] Siegel, A. F. (1978). Random arcs on the circle. *J. Appl. Prob.* **15**, 774-789.

[51] Siegel, A. F. and Holst, L. (1982). Covering the circle with random arcs of random sizes. *J. Appl. Prob.* **19**, 373-381.

[52] Silverman, B. and Brown, T. (1978). Short distances, flat triangles and Poisson limits. *J. Appl. Prob.* **15**, 816-826.

[53] Solomon, H. (1967). Random packing density. *Proc. Fifth Berkeley Symp. on Prob. and Stat.* **3**, 119-134, Univ. California Press.

[54] Solomon, H. and Sutton, C. (1986). A simulation study of random caps on a sphere. *J. Appl. Prob.* **23**, 951-960.

[55] Solomon, H. and Weiner, H. (1986). A review of the packing problem. *Commun. Statist. - Theor. Meth.* **15**, 2571-2607.

[56] Stein, C. (1987 seminar given at Stanford University).

[57] Stevens, W. L. (1939). Solution to a geometrical problem in probability. *Ann. Eugenics* **9**, 315-320.

[58] Thompson, T. M. (1983). *From Error-Correcting Codes Through Sphere Packing to Simple Groups*. The Mathematical Association of America.

[59] Wendel, J. G. (1962). A problem in geometric probability. *Math. Scand.* **2**, 109-111.

Appendix A: Packing by Hamming Distance

M is the number of words in a saturated packing of q-ary codewords of length n having minimum allowable interword distance d.

q	n	d	$E(M)$	$Var(M)$	# trials
2	3	2	3.493200	0.756829	10000
2	4	2	6.203800	2.815747	10000
2	5	2	11.080800	6.951967	10000
2	5	3	3.879800	0.225975	10000
2	6	2	20.171000	14.521211	10000
2	6	3	6.213600	0.704846	10000
2	6	4	3.363600	0.867882	10000
2	7	2	37.082400	27.182128	10000
2	7	3	9.938700	1.191661	10000
2	7	4	5.139200	1.673591	10000
2	8	2	68.899800	50.970457	10000
2	8	3	16.461900	0.758824	10000
2	8	4	8.237700	2.544253	10000
2	8	5	3.810200	0.343610	10000

2	9	2	129.102400	98.656380	10000
2	9	3	28.378700	1.705657	10000
2	9	4	13.091700	4.987190	10000
2	9	5	4.861000	1.020781	10000
2	9	6	3.288600	0.916802	10000
2	10	2	242.240800	168.855301	10000
2	10	3	49.495400	3.106489	10000
2	10	4	21.141000	10.550774	10000
2	10	5	7.588200	0.738095	10000
2	10	6	4.434000	0.916536	10000
2	11	2	457.204000	310.400785	1000
2	11	3	87.313000	5.254285	1000
2	11	4	34.859000	21.638758	1000
2	11	5	11.676000	0.655680	1000
2	11	6	6.417000	1.654766	1000
2	11	7	3.782000	0.388865	1000
2	12	2	867.509000	558.192111	1000
2	12	3	155.635000	9.639414	1000
2	12	4	58.842000	47.304340	1000
2	12	5	18.122000	0.956072	1000
2	12	6	9.853000	2.429821	1000
2	12	7	3.986000	0.027832	1000
2	12	8	3.310000	0.904805	1000

2	13	2	1653.710000	959.137273	100
2	13	3	279.360000	22.394343	100
2	13	4	98.880000	116.328889	100
2	13	5	28.770000	1.451616	100
2	13	6	15.040000	4.705455	100
2	13	7	6.060000	0.279192	100
2	13	8	3.960000	0.079192	100
2	14	2	3155.140000	1373.091313	100
2	14	3	504.080000	27.872323	100
2	14	4	170.270000	243.128384	100
2	14	5	46.490000	2.131212	100
2	14	6	22.720000	10.203636	100
2	14	7	9.050000	0.654040	100
2	14	8	5.040000	1.048889	100
2	14	9	3.660000	0.570101	100
2	15	2	6029.100000	3039.525253	100
2	15	3	918.180000	45.280404	100
2	15	4	296.880000	471.056162	100
2	15	5	75.600000	3.474747	100
2	15	6	35.450000	16.654040	100
2	15	7	13.810000	0.620101	100
2	15	8	7.720000	1.880404	100
2	15	9	4.000000	0.000000	100
2	15	10	3.200000	0.969697	100

2	16	2	11585.050000	5824.371212	100
2	16	3	1675.980000	104.302626	100
2	16	4	513.730000	1140.300101	100
2	16	5	124.400000	4.606061	100
2	16	6	55.270000	31.855657	100
2	16	7	20.340000	1.297374	100
2	16	8	11.640000	3.424646	100
2	16	9	5.300000	1.080808	100
2	16	10	3.920000	0.155152	100
2	17	2	22306.100000	14989.211111	10
2	17	3	3074.400000	291.155556	10
2	17	4	895.500000	3013.166667	10
2	17	5	207.600000	5.155556	10
2	17	6	86.600000	65.600000	10
2	17	7	29.700000	1.344444	10
2	17	8	17.300000	4.011111	10
2	17	9	7.000000	1.111111	10
2	17	10	4 200000	0.400000	10
2	17	11	3.400000	0.868966	30
2	18	5	345.700000	14.233333	10
2	18	6	147.500000	249.166667	10
2	18	7	45.700000	2.011111	10
2	18	8	25.700000	10.455556	10
2	18	9	10.300000	0.455556	10
2	18	10	6.600000	0.933333	10
2	18	11	4.000000	0.000000	10

2	19	9	15.800000	0.844444	10
2	19	10	8.500000	2.055556	10
2	19	11	5.000000	1.111111	10
2	20	10	13.800000	2.400000	10
2	20	11	6.400000	0.711111	10
3	3	2	7.474000	0.935792	2000
3	4	2	19.761500	2.621929	2000
3	4	3	6.222000	0.842137	2000
3	5	2	52.810333	8.612435	1500
3	5	3	11.976000	0.616732	2000
3	5	4	4.280000	0.908509	1000
3	6	2	142.937000	22.635667	1000
3	6	3	28.155333	1.626956	1500
3	6	4	9.509000	0.614534	1000
3	6	5	3.488000	0.250106	1000
3	7	2	392.005000	73.629899	200
3	7	3	68.813000	4.472504	1000
3	7	4	20.812000	1.492148	1000
3	7	5	7.384000	0.555099	1000
3	7	6	3.000000	0.000000	1000

3	8	2	1086.650000	225.603535	100
3	8	3	170.770000	9.593030	100
3	8	4	46.851000	2.705505	1000
3	8	5	15.295000	0.706682	1000
3	8	6	6.125000	0.169545	1000
3	8	7	3.000000	0.000000	1000
3	9	2	3038.200000	491.326316	20
3	9	3	435.040000	29.957980	100
3	9	4	108.394000	7.145910	1000
3	9	5	31.187000	1.667699	1000
3	9	6	11.759000	0.629549	1000
3	9	7	5.289000	0.780259	1000
3	10	4	253.100000	18.100000	10
3	10	5	66.700000	3.788889	10
3	10	6	22.900000	0.766667	10
3	10	7	9.100000	0.544444	10
3	11	6	46.600000	1.377778	10
3	11	7	17.700000	0.677778	10
4	3	2	13.390000	1.283221	4000
4	4	2	46.608000	4.988666	3000
4	4	3	11.035692	0.947948	3250

4	5	2	165.513333	19.055703	1200
4	5	3	30.464400	2.446511	2500
4	5	4	8.316000	0.431621	3000
4	6	2	597.522730	87.982516	220
4	6	3	91.980800	5.232618	1250
4	6	4	22.189091	1.263571	1100
4	7	2	2172.600000	527.410526	20
4	7	3	293.000000	18.621622	75
4	7	4	62.960000	3.412525	100
5	3	2	20.997500	1.506247	2000
5	4	2	91.401000	7.225424	1000
5	4	3	17.402000	0.945341	1000
5	5	2	405.015000	47.411839	200
5	5	3	62.226000	4.788501	500
5	5	4	14.246000	1.272757	1000
5	6	2	1823.800000	187.536842	20
5	6	3	233.180000	16.803673	50
5	6	4	43.940000	2.056970	100

Appendix B: Packing by Lee Distance

M denotes the number of words in a saturated packing of q-ary codewords of length n with minimum allowable interword distance d.

q	n	d	$E(M)$	$Var(M)$	# trials
4	3	2	20.074583	13.625545	2400
4	4	2	68.965417	54.404343	2400
4	5	2	242.035000	147.532607	400
4	6	2	869.071429	542.006763	140
4	7	2	3165.300000	2106.900000	10
4	4	3	16.467500	0.763686	4800
4	5	3	49.406250	3.092577	800
4	6	3	155.525000	10.228763	280
4	7	3	502.900000	18.936842	20

Cases having $n = 2$

d	q	$\hat{r}$	# trials
2	10	0.3652	1000
2	20	0.3642	1000
2	30	0.3631	500
2	40	0.3646	100
2	50	0.3644	100
2	55	0.3646	100
2	60	0.3645	100
3	20	0.1397	1000
3	30	0.1395	500
3	40	0.1396	200
3	50	0.1398	200
3	55	0.1399	100
3	60	0.1397	100
3	65	0.1398	100
3	70	0.1398	100
4	30	0.08027	1000
4	40	0.08032	500
4	50	0.08015	500
4	55	0.08023	200
4	60	0.08028	200
4	65	0.08020	100
4	70	0.08020	100
4	75	0.08034	100

5	40	0.04902	1000
5	50	0.04894	500
5	60	0.04917	200
5	70	0.04900	200
5	75	0.04915	200
5	80	0.04896	100
5	85	0.04898	100
5	90	0.04897	100
5	95	0.04908	100
6	50	0.03416	1000
6	60	0.03415	500
6	70	0.03418	200
6	80	0.03415	200
6	90	0.03417	100
6	95	0.03408	100
7	60	0.02448	100
7	70	0.02453	100
7	80	0.02451	100
7	90	0.02456	100
7	100	0.02452	100
8	80	0.01885	100
8	90	0.01881	100
8	100	0.01885	100
8	110	0.01882	100
8	120	0.01879	100

9	100	0.01467	100
9	105	0.01468	100
9	110	0.01466	100
9	115	0.01471	100
9	120	0.01462	100
9	125	0.01465	100
9	130	0.01467	200
10	115	0.01183	200
10	120	0.01189	200
10	125	0.01186	200
10	128	0.01188	200
10	130	0.01186	200
10	132	0.01187	200
11	120	0.009722	200
11	125	0.009705	200
11	128	0.009723	200
11	130	0.009733	200
11	132	0.009728	200
12	125	0.008162	100
12	126	0.008154	100
12	127	0.008188	100
12	128	0.008165	100
12	129	0.008204	100
12	130	0.008178	100
12	131	0.008161	100
12	132	0.008194	100

Appendix C: Packing Square Boxes

$\hat{\rho}$ is the average proportion of a q by q torus shaped lattice covered by a saturated packing of d by d blocks.

d	q	$\hat{\rho}$	# trials
2	23	0.7471	100
2	26	0.7433	100
2	29	0.7482	100
2	32	0.7496	100
2	35	0.7470	100
3	35	0.6786	100
3	40	0.6807	100
3	45	0.6785	100
3	50	0.6817	100
3	55	0.6774	100
4	60	0.6488	100
4	65	0.6498	100
4	70	0.6484	100
4	75	0.6479	100
4	80	0.6467	100
5	80	0.6328	100
5	90	0.6304	100
5	100	0.6280	100
5	110	0.6284	100
5	120	0.6307	100

6	105	0.6169	100
6	110	0.6200	100
6	115	0.6174	100
6	120	0.6176	100
6	125	0.6166	100
7	113	0.6097	100
7	116	0.6101	100
7	119	0.6087	100
7	122	0.6095	100
7	125	0.6080	100
8	112	0.6031	100
8	117	0.6032	100
8	122	0.6015	100
8	127	0.6038	100
8	132	0.6031	100
9	120	0.6017	100
9	123	0.5994	100
9	126	0.6005	100
9	129	0.6003	100
9	132	0.5976	100

Appendix D: Complementary Codes

n is the word length

$A(n, d)$ and $\mathcal{K}$ are defined in Section 2.6

max is the number of words in the largest randomly formed code

min is the number of words in the smallest randomly formed code

mean is the average number of words randomly packed

$\hat{r}$ is the observed center density

r^* is the observed center density for the corresponding case where the packing was done by the method of Section 2.2

n	$\mathcal{K} - \{n\}$	$A(n, d)$	max	min	mean	$\hat{r}$	r^*
4	2	8	8	8	8.000	0.500	0.388
5	2,3	16	10	8	9.344	0.292	0.346
6	3	8	4	4	4.000	0.063	0.097
6	2,4	32	32	32	32.000	0.500	0.315
6	2,3,4	32	32	12	21.868	0.342	0.315
7	3,4	16	16	16	16.000	0.125	0.078
7	2,5	64	14	8	12.805	0.100	0.290
7	2,3,4,5	64	44	16	35.875	0.280	0.290
8	4	16	16	16	16.000	0.063	0.032
8	3,5	20	4	4	4.000	0.016	0.064
8	3,4,5	20	16	16	16.000	0.063	0.064
8	2,6	128	16	8	14.616	0.057	0.269
8	2,4,6	128	128	128	128.000	0.500	0.269
8	2,3,5,6	128	26	16	23.784	0.093	0.269
8	2,3,4,5,6	128	128	52	69.631	0.272	0.269

9	4,5	20	16	16	16.000	0.031	0.026
9	3,6	40	8	8	8.000	0.016	0.055
9	3,4,5,6	40	32	20	24.995	0.049	0.055
9	2,7	256	18	8	16.522	0.032	0.252
9	2,3,6,7	256	18	8	17.006	0.033	0.252
9	2,4,5,7	256	74	32	50.684	0.099	0.252
9	2,3,4,5,6,7	256	158	98	128.318	0.251	0.252
10	5	12	4	4	4.000	0.004	0.007
10	4,6	40	32	20	24.998	0.024	0.021
10	4,5,6	40	28	4	9.620	0.009	0.021
10	3,7	[72,79]	4	4	4.000	0.004	0.048
10	3,5,7	[72,79]	4	4	4.000	0.004	0.048
10	3,4,6,7	[72,79]	32	16	25.475	0.025	0.048
10	3,4,5,6,7	[72,79]	52	42	47.260	0.046	0.048
10	2,8	512	20	8	18.457	0.018	0.237
10	2,3,7,8	512	20	8	18.234	0.018	0.237
10	2,5,8	512	20	8	17.540	0.017	0.237
10	2,4,6,8	512	512	512	512.000	0.500	0.237
10	2,3,5,7,8	512	40	16	35.031	0.034	0.237
10	2,3,4,6,7,8	512	512	32	316.110	0.309	0.237
10	2,3,5,6,8	512	512	36	316.583	0.309	0.237
10	2,3,4,5,6,7,8	512	348	212	240.520	0.235	0.237

11	5,6	24	24	8	23.760	0.012	0.006
11	4,7	[72,79]	16	10	15.784	0.008	0.017
11	4,5,6,7	[72,79]	32	24	27.620	0.013	0.017
11	3,8	[144,158]	4	4	4.000	0.002	0.043
11	3,4,7,8	[144,158]	32	32	32.000	0.016	0.043
11	3,5,6,8	[144,158]	18	12	15.280	0.007	0.043
11	3,4,5,6,7,8	[144,158]	104	84	92.060	0.045	0.043
11	2,9	1024	22	8	20.426	0.010	0.223
11	2,3,8,9	1024	22	8	20.064	0.010	0.223
11	2,4,7,9	1024	112	32	73.250	0.036	0.223
11	2,5,6,9	1024	24	8	20.921	0.010	0.223
11	2,3,4,7,8,9	1024	112	16	42.261	0.021	0.223
11	2,3,5,6,8,9	1024	58	22	35.415	0.017	0.223
11	2,4,5,6,7,9	1024	314	92	189.442	0.093	0.223
11	2,3,4,5,6,7,8,9	1024	530	426	460.100	0.225	0.223
12	6	24	24	8	23.680	0.006	0.002
12	5,7	32	4	4	4.000	0.001	0.004
12	5,6,7	32	24	18	21.810	0.005	0.004
12	4,8	[144,158]	32	32	32.000	0.008	0.014
12	4,5,7,8	[144,158]	64	20	34.020	0.008	0.014
12	4,6,8	[144,158]	104	80	91.950	0.022	0.014
12	4,5,6,7,8	[144,158]	96	40	62.000	0.015	0.014

12	3,9	256	4	4	4.000	0.001	0.038
12	3,4,8,9	256	32	12	25.120	0.006	0.038
12	3,6,9	256	24	16	20.520	0.005	0.038
12	3,5,7,9	256	4	4	4.000	0.001	0.038
12	3,4,6,8,9	256	106	12	78.620	0.019	0.038
12	3,4,5,7,8,9	256	64	44	58.606	0.014	0.038
12	3,5,6,7,9	256	48	30	45.090	0.011	0.038
12	3,4,5,6,7,8,9	256	168	146	157.632	0.038	0.038
12	2,10	2048	24	8	22.384	0.005	0.212
12	2,3,9,10	2048	24	8	22.060	0.005	0.212
12	2,6,10	2048	24	16	22.440	0.005	0.212
12	2,4,8,10	2048	134	20	58.040	0.014	0.212
12	2,3,6,9,10	2048	26	8	22.544	0.006	0.212
12	2,3,4,8,9,10	2048	134	16	57.173	0.014	0.212
12	2,5,7,10	2048	38	16	34.690	0.008	0.212
12	2,4,6,8,10	2048	2048	2048	2048.000	0.500	0.212
12	2,3,5,7,9,10	2048	48	16	43.473	0.011	0.212
12	2,3,4,6,8,9,10	2048	2048	30	1683.946	0.411	0.212
12	2,4,5,7,8,10	2048	116	30	59.549	0.015	0.212
12	2,5,6,7,10	2048	36	20	27.690	0.007	0.212
12	2,3,4,5,7,8,9,10	2048	240	46	110.170	0.027	0.212
12	2,3,5,6,7,9,10	2048	48	32	43.940	0.011	0.212
12	2,4,5,6,7,8,10	2048	2048	128	823.857	0.201	0.212
12	2,3,4,5,6,7,8,9,10	2048	1120	796	867.528	0.212	0.212

Densest packing schemes

n	d	$\mathcal{K}-\{n\}$	$\hat{r}$	r^*
4	2	2	0.500	0.388
5	2	2,3	0.292	0.346
6	3	3	0.063	0.097
6	2	2,4	0.500	0.315
7	3	3,4	0.125	0.078
7	2	2,3,4,5	0.280	0.290
8	4	4	0.063	0.032
8	3	3,4,5	0.063	0.064
8	2	2,4,6	0.500	0.269
9	4	4,5	0.031	0.026
9	3	3,4,5,6	0.049	0.055
9	2	2,3,4,5,6,7	0.251	0.252
10	5	5	0.004	0.007
10	4	4,6	0.024	0.021
10	3	3,4,5,6,7	0.046	0.048
10	2	2,4,6,8	0.500	0.237
11	5	5,6	0.012	0.006
11	4	4,5,6,7	0.013	0.017
11	3	3,4,5,6,7,8	0.045	0.043
11	2	2,3,4,5,6,7,8,9	0.225	0.223
12	6	6	0.006	0.002
12	5	5,6,7	0.005	0.004
12	4	4,6,8	0.022	0.014
12	3	3,4,5,6,7,8,9	0.038	0.038
12	2	2,4,6,8,10	0.500	0.212

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20. ABSTRACT

In a sequential packing problem, random objects are uniformly and independently selected from some space. A selected object is either packed or rejected, depending on the distance between it and the nearest object which has been previously packed. A saturated packing is said to exist when it is no longer possible to pack any additional selections. The random packing density is the average proportion of the space which is occupied by the packed objects at saturation.

Results concerning the time of the first rejection in a packing sequence are given in Chapter 1. The accuracy of some common approximation formulas is investigated for several settings. The problems considered may be thought of as generalizations of the classical birthday problem.

Exact results concerning random packing densities are generally known only for some packing sequences in one-dimensional spaces. In Chapter 2, the packing densities of various computer generated codes are examined. These stochastically constructed codes provide a convenient way to study packing in multidimensional spaces. Asymptotic approximation formulas are given for the packing densities which arise from several different coding schemes. In one special case considered, a new method is found for approximating a planar density. The result obtained agrees closely with estimates obtained by others.

In Chapter 3 the distribution of the number of random selections needed to achieve a saturated packing is considered. In each of the settings examined, the results are compared with analogous results from an associated random covering problem.

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